YILDIZ TECHNICAL UNIVERSITY
MECHANICAL FACULTY / MECHATRONICS ENGINEERING DEPARTMENT

Student Name:	**Student number:**	**Signature:**
Course: MKT3841 Numerical Methods Gr2	**Date/Time:** 26.12.2019 / 15:00	**Duration:** 90 minutes
Exam Type: Final Exam	**Course Instructor:** Assoc.Prof. Cüneyt Yılmaz	

Question 1) [25 points] Apply the finite difference method in the following differential equation of a heat transfer model to obtain a simultaneous equation system in the matrix form $\boldsymbol{A.T = B}$ where $\boldsymbol{T} = [\boldsymbol{T_1}, \boldsymbol{T_2}, \cdots, \boldsymbol{T_n}]$.

$\frac{d^2T}{dx^2} = -0.5 \cdot (T - 20)$, with $T(0) = 40$, $T'(4) = 50$ and $h = \Delta x = 1$.

	$f(x-2h)$	$f(x-h)$	$f(x)$	$f(x+h)$	$f(x+2h)$
$2hf'(x)$		−1	0	1	
$h^2 f''(x)$		1	−2	1	
$2h^3 f'''(x)$	−1	2	0	−2	1
$h^4 f^{(4)}(x)$	1	−4	6	−4	1

Coefficients of central finite difference approximations

	$f(x-4h)$	$f(x-3h)$	$f(x-2h)$	$f(x-h)$	$f(x)$
$hf'(x)$				−1	1
$h^2 f''(x)$			1	−2	1
$h^3 f'''(x)$		−1	3	−3	1
$h^4 f^{(4)}(x)$	1	−4	6	−4	1

Coefficients of backward finite difference approximations of $\mathcal{O}(h)$

	$f(x)$	$f(x+h)$	$f(x+2h)$	$f(x+3h)$	$f(x+4h)$
$hf'(x)$	−1	1			
$h^2 f''(x)$	1	−2	1		
$h^3 f'''(x)$	−1	3	−3	1	
$h^4 f^{(4)}(x)$	1	−4	6	−4	1

Coefficients of forward finite difference approximations of $\mathcal{O}(h)$

Solution 1)

$$n = \frac{x(n)-x(0)}{h} + 1 = \frac{4-0}{1} + 1 = 5$$

$$T_1 = 40 \, , \, T_5' = 50 \quad and \quad h = 1$$

$$T_i' = \left(\frac{1}{2h}\right)(T_{i+1} - T_{i-1}) = 0.5 \cdot (T_{i+1} - T_{i-1})$$

$$T_i'' = \left(\frac{1}{h^2}\right)(T_{i-1} - 2\,T_i + T_{i+1}) = T_{i-1} - 2\,T_i + T_{i+1}$$

$$T_{i-1} - 2\,T_i + T_{i+1} + 0.5 \cdot (T_i - 20) = 0$$

$$T_{i-1} - 1.5\,T_i + T_{i+1} = 10$$

i=1: $\boldsymbol{x_1 = 0}$, $\boldsymbol{T_1 = 40}$

i=2: $\boldsymbol{x_2 = 1}$, $\boldsymbol{T_1 - 1.5\,T_2 + T_3 = 10}$

i=3: $\boldsymbol{x_2 = 2}$, $\boldsymbol{T_2 - 1.5\,T_3 + T_4 = 10}$

i=4: $\boldsymbol{x_2 = 3}$, $\boldsymbol{T_3 - 1.5\,T_4 + T_5 = 10}$

i=5: $\boldsymbol{x_2 = 4}$, $\boldsymbol{T_4 - 1.5\,T_5 + T_6 = 10}$

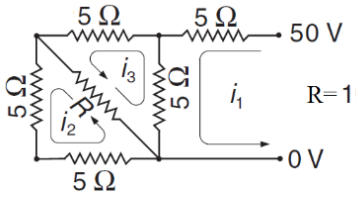
$$T_5' = 0.5\,(T_6 - T_4) = 50 \;\rightarrow\; T_6 = 100 + T_4$$

$$T_4 - 1.5\,T_5 + (100 + T_4) = 10 \quad \rightarrow \quad \boldsymbol{2 \cdot T_4 - 1.5\,T_5 = -90}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1.5 & 1 & 0 & 0 \\ 0 & 1 & -1.5 & 1 & 0 \\ 0 & 0 & 1 & -1.5 & 1 \\ 0 & 0 & 0 & 2 & -1.5 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 40 \\ 10 \\ 10 \\ 10 \\ -90 \end{bmatrix}$$

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Question 2) [25 points] Consider the following electrical circuit.



$$5i_1 + 5(i_1 - i_3) = 50\ \text{V}$$

$$10(i_2 - i_3) + 5i_2 + 5i_2 = 0$$

$$5i_3 + 10(i_3 - i_2) + 5(i_3 - i_1) = 0$$

Rewrite the system in the matrix form as $\boldsymbol{A \cdot X = B}$, and solve $\{ i_1 , i_2 , i_3 \}^T$ by using Doolittle's LU Decomposition method.

Solution 2)

$$\begin{bmatrix} 10 & 0 & -5 \\ 0 & 20 & -10 \\ -5 & -10 & 20 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A} = \begin{bmatrix} 10 & 0 & -5 \\ 0 & 20 & -10 \\ -5 & -10 & 20 \end{bmatrix} \quad , \quad \mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{A = L \cdot U}$$

First, we use Gauss elimination:

row2 – (0) x row1 → row2 , $L_{21} = 0$ and,

row3 – (-1/2) x row1 → row3 , $L_{31} = -1/2$ → $\boldsymbol{A'} = \begin{bmatrix} 10 & 0 & -5 \\ 0 & 20 & -10 \\ 0 & -10 & 17.5 \end{bmatrix}$

row3 – (-1/2) x row2 → row3 , $L_{32} = -1/2$ → $\boldsymbol{U} = \begin{bmatrix} 10 & 0 & -5 \\ 0 & 20 & -10 \\ 0 & 0 & 12.5 \end{bmatrix}$ and $\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$

$$\boldsymbol{A \cdot X = B}$$

$$\boldsymbol{L \cdot (U \cdot X) = B}$$

$$\boldsymbol{Y = U \cdot X} \quad \rightarrow \quad \boldsymbol{L \cdot Y = B} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \cdot \mathbf{Y} = \begin{bmatrix} 50 \\ 0 \\ 0 \end{bmatrix}$$

$y_1 = 50$, $y_2 = 0$, $y_3 = 25$ → $\mathbf{Y} = \begin{bmatrix} 50 \\ 0 \\ 25 \end{bmatrix}$

$$\boldsymbol{U \cdot X = Y} \quad \rightarrow \quad \begin{bmatrix} 10 & 0 & -5 \\ 0 & 20 & -10 \\ 0 & 0 & 12.5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 50 \\ 0 \\ 25 \end{bmatrix} \quad \rightarrow$$

$x_3 = \frac{25}{12.5} = 2$, $x_2 = \frac{0+10\cdot(2)}{-20} = 1$, $x_1 = \frac{50-0\cdot(1)+5\cdot(2)}{10} = 6$ → $\mathbf{X} = \begin{Bmatrix} \boldsymbol{i_1} \\ \boldsymbol{i_2} \\ \boldsymbol{i_3} \end{Bmatrix} = \begin{Bmatrix} \mathbf{6} \\ \mathbf{1} \\ \mathbf{2} \end{Bmatrix}$ ***ampers***

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Question 3) **[25 points]** Use the data below to compute the speed and the accelaration at *t=0.2 sec* as accurately as possible.

Time (sec)	*0*	*0.1*	*0.2*	*0.3*	*0.4*
Position (meter)	*0.000*	*0.080*	*0.140*	*0.300*	*0.750*

Hint: Use the table for the first central difference approximation of O(h^2) and Richardson Extrapolation equation to increase the accuracy: $G = \frac{2^P g(h_1/2) - g(h_1)}{2^P - 1}$

Work with at least 3 decimal points.

	$f(x-2h)$	$f(x-h)$	$f(x)$	$f(x+h)$	$f(x+2h)$
$2hf'(x)$		-1	0	1	
$h^2 f''(x)$		1	-2	1	
$2h^3 f'''(x)$	-1	2	0	-2	1
$h^4 f^{(4)}(x)$	1	-4	6	-4	1

Solution 3) *p=2*

The Speed:

$f'(x) = (\frac{1}{2h})\,[\, f(x+h) - f(x-h)\,]$

for h=0.1 , $f'(x) = \left(\frac{1}{2(0.1)}\right)[\, 0.300 - 0.080\,] = 1.1$ → g(h_1/2)

for h=0.2 , $f'(x) = \left(\frac{1}{2(0.2)}\right)[\, 0.750 - 0.000] = 1.875$ → g(h_1)

$G = \frac{2^p g\left(\frac{h_1}{2}\right) - g(h_1)}{2^p - 1} = \frac{2^2(1.1) - 1.875}{2^2 - 1} = \mathbf{0.842\ m/s}$ with the error O(h^4).

The Accelaration:

$f''(x) = (1/h^2)\,[\, f(x-h) - 2\,f(x) + f(x+h)\,]$

for h=0.1 ,

$f''(0.1) = \left(\frac{1}{(0.1)^2}\right)[\, (0.080) - 2\,(0.140) + (0.300)] = 10.000$ → g(h_1/2)

for h=0.2 ,

$f''(0.2) = \left(\frac{1}{(0.2)^2}\right)[\, (0.000) - 2\,(0.140) + (0.750)] = 11.750$ → g(h_1)

f "(0.2)=(1/0.2^2) (0.000-2·0.140+0.750) = 11.750 → g(h_1)

$G = \frac{2^2(10.000) - 11.750}{2^2 - 1} = \mathbf{9.417\ m/s^2}$ with the error O(h^4).

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Question 4) [25 points] The volume of a closed cylinder is given with a formula $f(x) = \frac{S}{2} \cdot x - \pi \cdot x^3$ where x is its radius, and S is its surface area. If we want to keep the surface area at $150\ cm^2$, then numerically calculate its **radius x** to obtain the maximum volume $f_{max}(x)$ by using **Golden Bracketing** method.

(Search x between 2.7 and $2.9\ cm$. Stop telescoping when the bracketing width $h \leq 1 \cdot 10^{-1}$. Use $R=0.618034$ as the Golden Ratio. Work with at least 4 decimal points.)

Solution 4) To convert the problem from maximization to minimization, we negate the function as:

$$F(x) = -f(x) = -75 \cdot x + \pi \cdot x^3$$

and numerically minimize it by using Golden Bracketing method as:

i	a_i	b_i	h	$a_i'=b_i-R_xh$	$b_i'=a_i+R_xh$	$f(a_i')$	$f(b_i')$	Stop if $h<\varepsilon$
1	2.7000	2.9000	0.2000	2.7764	2.8236	-140.9949	-141.0472	
2	2.7764	2.9000	0.1236	2.8236	2.8528	-141.0472	-141.0203	
3	2.7764	2.8528	0.0764	2.8056	2.8236	-141.0411	-141.0472	Yes

$$\boldsymbol{xmin = \frac{2.8056 + 2.8236}{2} = 2.8146} \qquad \boldsymbol{F(xmin) = -141.0463}$$

Calculation details at the first and the last step:

At the 1st step (i=1):

$$f(a_1') = f(2.7764) = -75 \cdot (2.7764) + \pi \cdot (2.7764)^3 = -140.9949$$

$$f(b_1') = f(2.82236) = -75 \cdot (2.82236) + \pi \cdot (2.82236)^3 = -141.0472$$

$$f(a_1') > f(b_1') \quad \rightarrow \quad a_2 = a_1' = 2.7764 \quad and \quad b_2 = b_1 = 2.9000$$

At the 3rd step (i=3): $h = b_3 - a_3 = 2.8525 - 2.7764 = 0.0764 < 0.1$, the telescoping stopped.

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