YILDIZ TECHNICAL UNIVERSITY
MECHANICAL FACULTY / MECHATRONICS ENGINEERING DEPARTMENT

Student Name: SOLUTIONS	**Number:**	**Signature:**
Course: MKT3841 Numerical Methods	**Date:** 22.01.2025	**Duration:** 75 min.
Exam Type: Final Exam	**Course Instructor:** Prof.Dr. Cüneyt Yılmaz	

Question 1) [33 Points] Obtain the finite difference equations in the matrix form for the following differential equation from x=0 to 3 for n=4:

$\frac{d^2y}{dx^2} = -\frac{dy}{dx} + 5x$ with $y'(0) = 1$ and $y(3) = 2$.

	$f(x-2h)$	$f(x-h)$	$f(x)$	$f(x+h)$	$f(x+2h)$
$2hf'(x)$		−1	0	1	
$h^2 f''(x)$		1	−2	1	
$2h^3 f'''(x)$	−1	2	0	−2	1
$h^4 f^{(4)}(x)$	1	−4	6	−4	1

Solution 1) *h=Δx=(3-0)/(n-1)= (3-0)/(4-1) = 1*

Write the differential equation in the finite difference form (with the index variable i) by using the table:

From the table: $y_i' = [1/(2h)](y_{i+1} - y_{i-1}) = 0.5\,(y_{i+1} - y_{i-1})$

$y_i'' = (1/h^2)(y_{i-1} - 2y_i + y_{i+1}) = y_{i-1} - 2y_i + y_{i+1}$

Diff. Eq.: $y_i'' = -y_i' + 5x_i \rightarrow (y_{i-1} - 2y_i + y_{i+1}) + 0.5\,(y_{i+1} - y_{i-1}) = 5x_i$

$0.5\,y_{i-1} - 2y_i + 1.5\,y_{i+1} = 5x_i$, *i=1 to 3 (because we have y_4 available)*

Write x_i and the finite difference equations for each step (i):

i=1: $x_1 = 0$, $0.5\,y_0 - 2y_1 + 1.5\,y_2 = 5x_1$

$y'_1 = 1 \rightarrow y'_1 = (y_2 - y_0)/(2h) = (y_2 - y_0)/2 = 1 \rightarrow y_0 = y_2 - 2$

$0.5\,(y_2 - 2) - 2y_1 + 1.5\,y_2 = 0 \rightarrow -2y_1 + 2y_2 = 1$

i=2: $x_2 = 1$, $0.5\,y_1 - 2y_2 + 1.5\,y_3 = 5x_2 \rightarrow 0.5\,y_1 - 2y_2 + 1.5\,y_3 = 5$

i=3: $x_3 = 2$, $0.5\,y_2 - 2y_3 + 1.5\,y_4 = 5x_3 \rightarrow 0.5\,y_2 - 2y_3 + 1.5\,y_4 = 10$

i=4: $x_4 = 3$, $y_4 = 2$

Write the finite difference equations in the matrix form:

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0.5 & -2 & 1.5 & 0 \\ 0 & 0.5 & -2 & 1.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \\ 2 \end{bmatrix}$$

tonozdayim.com/@seventhgrade

Question 2) [33 Points] Use polynomial interpolation to compute $f'(2)$ and $f''(2)$ for the data set below.

x	5	-7	-6	0
f(x)	1	-23	-54	-954

Hint: Polynomials in Newton Form

$$P_n(x) = P_{n-1}(x) + a_n \prod_{j=0}^{n-1}(x - x_j)$$

where *n* is the degree of polynomial.

Solution 2) Write the equations in Newton form:

$P_0(x) = a_0$

$P_1(x) = a_0 + a_1(x - 5)$

$P_2(x) = a_0 + a_1(x - 5) + a_2(x - 5)(x + 7)$

$\boldsymbol{P_3(x) = a_0 + a_1(x - 5) + a_2(x - 5)(x + 7) + a_3(x - 5)(x + 7)(x + 6)}$

Solve the parameters of the Newton Polynomial:

$P_0(5) = f(5) = 1 \quad \rightarrow \quad \boldsymbol{a_0 = 1}$

$P_1(-7) = f(-7) = -23 = 1 + a_1(-7 - 5) \quad \rightarrow \quad \boldsymbol{a_1 = \frac{-24}{-12} = 2}$

$P_2(-6) = f(-6) = -54 = 1 + 2(-6 - 5) + a_2(-6 - 5)(-6 + 7) \quad \rightarrow \quad \boldsymbol{a_2 = \frac{-33}{-11} = 3}$

$P_3(0) = f(0) = -954 = 1 + 2(0 - 5) + 3(0 - 5)(0 + 7) + a_3(0 - 5)(0 + 7)(0 + 6)$

$$\rightarrow \quad \boldsymbol{a_3 = \frac{-840}{-210} = 4}$$

$P_3(x) = 1 + 2(x - 5) + 3(x - 5)(x + 7) + 4(x - 5)(x + 7)(x + 6)$

$P_3(x) = 1 + 2x - 10 + 3(x^2 + 2x - 35) + (4x - 20)(x^2 + 13x + 42)$

$P_3(x) = 1 + 2x - 10 + 3x^2 + 6x - 105 + 4x^3 + 52x^2 + 168x - 20x^2 - 260x - 840$

$\boldsymbol{P_3(x) = f(x) = 4x^3 + 35x^2 - 84x - 954}$

Calculate the derivatives for x=2:

a) $f'(x) = 12x^2 + 70x - 84$

$\boldsymbol{f'(2)} = 12(2)^2 + 70(2) - 84 = \mathbf{104}$

b) $f''(x) = 24x + 70$

$\boldsymbol{f''(2)} = 24(2) + 70 = \mathbf{118}$

tonozdayim.com/@seventhgrade

Question 3) [34 Points] A plane has a variable speed (in meter/seconds) with respect to time t (seconds) as $v(t) = 5 + 33\,t - 7\,t^2$. By using **Newton Raphson** method (with numerical derivatives by 1st central finite difference approach), numerically calculate the maximum speed of this plane. In your solution, apply the followings:

- Write the function to optimize and add a constraint function considering a positive t value. Assume $\lambda=1$.
- Start the iteration at $t = 1$ *second*.
- Use $h = 0.1$ for numerical derivatives.
- Stop the iteration when $|t_{i+1} - t_i| < 0.01$.
- Show your detailed calculations on the next page.
- Work with at least 3 decimal places where available.

Hint: Newton Raphson formula to numerically find the root of $f(x)=0$ is $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$.

Solution 3) Prepare the function for optimization:

To convert the problem from maximization to minimization, we negate the given function v(t). We also add the constraint function $[\min(0,t)]^2$ to ensure positive t values.

$$f(t) = -5 - 33\,t + 7\,t^2 + [\min(0,t)]^2$$

Newton Raphson Formulation for optimization: $t_{i+1} = t_i - \frac{f'(t_i)}{f''(t_i)}$

where $f'(t_i) = \frac{1}{2h}[f(t_i+h) - f(t_i-h)]$ and $f''(t_i) = \frac{1}{h^2}[f(t_i+h) - 2\,f(t_i) + f(t_i-h)]$

and numerically minimize it by using Newton-Raphson method as:

i	h	t_i t_i-h t_i+h	$f(t_i)$	$f'(t_i)$	$f''(t_i)$	t_{i+1}	$\lvert t_{i+1}-t_i\rvert$
1	0.100	1.000	-31.000	-19.000	14.000	2.357	1.357
		0.900	-29.030				
		1.100	-32.830				
2	0.100	2.357	-43.893	0.000	14.000	2.357	0.000
		2.257	-43.823				
		2.457	-43.823				

$$at \quad t = 2.357\ s \quad , \quad v_{max} = 43.893\ m/s$$

tonozdayim.com/@seventhgrade