YILDIZ TECHNICAL UNIVERSITY
MECHANICAL FACULTY / MECHATRONICS ENGINEERING DEPARTMENT

Student Name:	**Number:**	**Signature:**
Course: MKT3841 Numerical Methods Gr1&2	**Date:** 11.01.2023 **Time:** 18:00	
Exam Type: Final Exam	**Course Instructor:** Assoc.Prof. Cüneyt Yılmaz	

Question 1) [33 Points] Obtain the finite difference equations in the matrix form for the following differential equation from x=0 to 3 for n=4:

$\frac{d^2y}{dx^2} = -\frac{dy}{dx} + 5x$ with $y'(0) = 1$ and $y(3) = 2$.

	$f(x-2h)$	$f(x-h)$	$f(x)$	$f(x+h)$	$f(x+2h)$
$2hf'(x)$		−1	0	1	
$h^2 f''(x)$		1	−2	1	
$2h^3 f'''(x)$	−1	2	0	−2	1
$h^4 f^{(4)}(x)$	1	−4	6	−4	1

Solution 1) Do the necessary calculations and write the differential equation in the finite difference form (with the index variable i) by using the given table:

h=Δx=(3-0)/(n-1)= (3-0)/(4-1) = 1

From the table: $y_i' = [1/(2h)]\,(y_{i+1} - y_{i-1}) = 0.5\,(y_{i+1} - y_{i-1})$

$y_i'' = (1/h^2)\,(y_{i-1} - 2y_i + y_{i+1}) = y_{i-1} - 2y_i + y_{i+1}$

Diff. Eq.: $y_i'' = -y_i' + 5\,x_i \rightarrow (y_{i-1} - 2y_i + y_{i+1}) + 0.5\,(y_{i+1} - y_{i-1}) = 5\,x_i$

$0.5\,y_{i-1} - 2\,y_i + 1.5\,y_{i+1} = 5\,x_i$, *i=1 to 3 (because we have y_4 available)*

Write the finite difference equations (***x_i*** and the ***finite difference equations***) for each step (***i***):

i=1: $x_1 = 0$, $0.5\,y_0 - 2\,y_1 + 1.5\,y_2 = 5\,x_1$

$y'_1 = 1 \rightarrow y'_1 = (y_2 - y_0)/(2h) = (y_2 - y_0)/2 = 1 \rightarrow y_0 = y_2 - 2$

$0.5\,(y_2 - 2) - 2\,y_1 + 1.5\,y_2 = 0 \rightarrow -2\,y_1 + 2\,y_2 = 1$

i=2: $x_2 = 1$, $0.5\,y_1 - 2\,y_2 + 1.5\,y_3 = 5\,x_2 \rightarrow 0.5\,y_1 - 2\,y_2 + 1.5\,y_3 = 5$

i=3: $x_3 = 2$, $0.5\,y_2 - 2\,y_3 + 1.5\,y_4 = 5\,x_3 \rightarrow 0.5\,y_2 - 2\,y_3 + 1.5\,y_4 = 10$

i=4: $x_4 = 3$, $y_4 = 2$

Write the finite difference equations in the matrix form:

$$\begin{bmatrix} -2 & 2 & 0 & 0 \\ 0.5 & -2 & 1.5 & 0 \\ 0 & 0.5 & -2 & 1.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 10 \\ 2 \end{bmatrix}$$

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Question 2) [33 Points] Solve the simultaneous equation system below by using **Gauss Elimination Method** with all details (clearly showing forward and backward calculations. No small street market calculations.).

$$6\,x_2 + 15\,x_3 - 150 = 0$$
$$-3\,x_1 - 9\,x_2 + 6\,x_3 - 9 = 0$$
$$9\,x_1 - 9\,x_2 + 2\,x_3 - 25 = 0$$

Solution 2) You have to switch the 1st row with one of the others.

Swtich the 1st and 2nd rows:

$-3\ \ x_1 - 9\,x_2 + 6\,x_3 = 9$
$0\ \ x_1 + 6\,x_2 + 15\,x_3 = 150$
$9\ \ x_1 - 9\,x_2 + 2\,x_3 = 25$

Gauss Elimination

Row2 + 0 x Row1 → Row2 & Row3 + 3 x Row1 → Row3 :

$-3\,x_1 - 9\,x_2 + 6\,x_3 = 9$
$0\,x_1 + 6\,x_2 + 15\,x_3 = 150$
$0\,x_1 - 36\,x_2 + 20\,x_3 = 52$

Row3 + 6 x Row2 → Row3 :

$-3\,x_1 - 9\,x_2 + 6\,x_3 = 9$
$0\,x_1 + 6\,x_2 + 15\,x_3 = 150$
$0\,x_1 + 0\,x_2 + 110\,x_3 = 952$

Backward calculation:

$x_3 = 952 / 110 \rightarrow x_3 = 8.654545$
$x_2 = (150 - 15 \cdot 8.654545)/6 \rightarrow x_2 = 3.364$
$x_1 = -(9 + 9 \cdot 3.363636 - 6 \cdot 8.7)/3 \rightarrow x_1 = 4.2182$

Sorry for fractional results ☺

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Question 3) [34 Points] The volume of a closed cylinder is given with a formula $f(x) = \frac{S}{2} \cdot x - \pi \cdot x^3$

where x is its radius, and S is its surface area. If we want to keep the surface area at $150\ cm^2$, then numerically calculate its **radius x** to obtain the maximum volume $f_{max}(x)$ by using **Golden Bracketing** method. Consider the constraint that x should be greater than zero.

(Search x between 2.7 and $2.9\ cm$. Stop telescoping when the bracketing width $h \leq 1 \cdot 10^{-1}$. Use $R=0.618034$ as the Golden Ratio. Work with at least 4 decimal points.)

Solution 3) Prepare the function to do the optimization:

To convert the problem from maximization to minimization, we negate the function as:

$$F(x) = -f(x) = -75 \cdot x + \pi \cdot x^3 + \lambda \cdot [\min(0, x)]^2 \qquad (Let\ \lambda = 1)$$

and use the table below to numerically optimize the function by using Golden Bracketing method:

(Do not expect more rows than given below)

i	a_i	b_i	h	$x_{1i}=b_i-R_xh$	$x_{2i}=a_i+R_xh$	$f(x_{1i})$	$f(x_{2i})$	Stop if $h<\varepsilon$
1	2.7000	2.9000	0.2000	2.7764	2.8236	-140.9949	-141.0472	
2	2.7764	2.9000	0.1236	2.8236	2.8528	-141.0472	-141.0203	
3	2.7764	2.8528	0.0764	2.8056	2.8236	-141.0411	-141.0472	Yes

Show the detailed calculation for the last row only:

At the 3rd step (i=3):

$$f(x_{22}) > f(x_{12}) \quad \rightarrow \quad a_3 = a_2 \ \ and \ \ b_3 = x_{22} = 2.8528$$

$$f(x_{13}) = f(2.8056) = -75 \cdot (2.8056) + \pi \cdot (2.8056)^3 = -141.0411$$

$$f(x_{23}) = f(2.8236) = -75 \cdot (2.8236) + \pi \cdot (2.8236)^3 = -141.0472$$

At the 3rd step (i=3): $h = b_3 - a_3 = 2.8525 - 2.7764 = 0.0764 < 0.1$, the telescoping stopped.

Optimal x=

$$\boldsymbol{Optimal\ \ x = \frac{2.8056 + 2.8236}{2} = 2.8146\ cm}$$

Maximum volume=

$$\boldsymbol{V_{max} = 75 \cdot (2.8146) - \pi \cdot (2.8146)^3 = 141.0463\ cm^3}$$

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