YILDIZ TECHNICAL UNIVERSITY
MECHANICAL FACULTY / MECHATRONICS ENGINEERING DEPARTMENT

Student Name:	**Student number:**	**Signature:**
Course: MKT3841 Numerical Methods	**Date/Time:** 21.05.2019 / 13:00	**Duration:** 90 minutes
Exam Type: Final Exam	**Course Instructor:** Assoc.Prof. Cüneyt Yılmaz	

Question 1) [25 points] Apply the finite difference method in the following differential equation to obtain a simultaneous equation system in the matrix form $\boldsymbol{A}.\boldsymbol{y} = \boldsymbol{B}$ where $\boldsymbol{y} = [\boldsymbol{y_1}\ ,\ \boldsymbol{y_2}\ ,\ \cdots\ ,\ \boldsymbol{y_n}]$.

$\frac{d^2y}{dt^2} = 2\ \frac{dy}{dt} + 2\,y + e^{-t}$, with $y(0) = 0\ ,\ y'(1) = -1/(12e)$ and $\Delta t = 0.25$.

	$f(x-2h)$	$f(x-h)$	$f(x)$	$f(x+h)$	$f(x+2h)$
$2hf'(x)$		−1	0	1	
$h^2f''(x)$		1	−2	1	
$2h^3f'''(x)$	−1	2	0	−2	1
$h^4f^{(4)}(x)$	1	−4	6	−4	1

Coefficients of central finite difference approximations

	$f(x-4h)$	$f(x-3h)$	$f(x-2h)$	$f(x-h)$	$f(x)$
$hf'(x)$				−1	1
$h^2f''(x)$			1	−2	1
$h^3f'''(x)$		−1	3	−3	1
$h^4f^{(4)}(x)$	1	−4	6	−4	1

Coefficients of backward finite difference approximations of $\mathcal{O}(h)$

	$f(x)$	$f(x+h)$	$f(x+2h)$	$f(x+3h)$	$f(x+4h)$
$hf'(x)$	−1	1			
$h^2f''(x)$	1	−2	1		
$h^3f'''(x)$	−1	3	−3	1	
$h^4f^{(4)}(x)$	1	−4	6	−4	1

Coefficients of forward finite difference approximations of $\mathcal{O}(h)$

Solution 1) $n = \frac{t(n)-t(0)}{h} + 1 = \frac{1.00-0}{0.25} + 1 = 5$

$$y_1 = 0 \quad and \quad y_4' = \frac{-1}{12e}$$

$$y_i' = \left(\frac{1}{2h}\right)(\,y_{i+1} - y_{i-1}\,) = 2\,(\,y_{i+1} - y_{i-1}\,)$$

$$y_i'' = \left(\frac{1}{h^2}\right)(\,y_{i-1} - 2\,y_i + y_{i+1}\,) = 16\,(\,y_{i-1} - 2\,y_i + y_{i+1}\,)$$

$$16\,(\,y_{i-1} - 2\,y_i + y_{i+1}\,) - 2\cdot 2\,(\,y_{i+1} - y_{i-1}\,) - 2\,y_i = e^{-t_i}$$

$$20\,y_{i-1} - 34\,y_i + 12\,y_{i+1} = e^{-t_i}$$

i=1: $\boldsymbol{t_1 = 0.00}$, $\boldsymbol{y_1 = 0}$

i=2: $\boldsymbol{t_2 = 0.25}$, $\boldsymbol{20\,y_1 - 34\,y_2 + 12\,y_3 = e^{-0.25}}$

i=3: $\boldsymbol{t_3 = 0.50}$, $\boldsymbol{20\,y_2 - 34\,y_3 + 12\,y_4 = e^{-0.50}}$

i=4: $\boldsymbol{t_4 = 0.75}$, $\boldsymbol{20\,y_3 - 34\,y_4 + 12\,y_5 = e^{-0.75}}$

i=5: $\boldsymbol{t_5 = 1.00}$, $20\,y_4 - 34\,y_5 + 12\,y_6 = e^{-1.00}$

$$y_5' = 2\,(\,y_6 - y_4\,) = \frac{-1}{12e} \rightarrow y_6 = \frac{-1}{12e} + y_4$$

$$20\,y_4 - 34\,y_5 + 12\left(\frac{-1}{12e} + y_4\right) = \frac{1}{e} \qquad \rightarrow \qquad \boldsymbol{32\,y_4 - 34\,y_5 = 2/e}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 20 & -34 & 12 & 0 & 0 \\ 0 & 20 & -34 & 12 & 0 \\ 0 & 0 & 20 & -34 & 12 \\ 0 & 0 & 0 & 32 & -34 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-0.25} \\ e^{-0.50} \\ e^{-0.75} \\ 2/e \end{bmatrix}$$

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Question 2) [25 points] Determine the coefficients of the polynomial $y = a_0 + a_1\,x + a_2\,x^2$ that passes through the (x, y) points $(1, 6), (3, 18)$ and $(5, 38)$. Rewrite the system in the matrix form as $\boldsymbol{A \cdot X = B}$, and solve $\{\, a0\,, a1\,, a2\,\}^T$ by using Doolittle's LU Decomposition method.

Solution 2) $y = a_0 + a_1\,x + a_2\,x^2$

$x = 1\ and\ y = 6 \quad \rightarrow \quad 6 = a_0 + a_1 + a_2$ $x = 3\ and\ y = 18 \quad \rightarrow \quad 18 = a_0 + 3\,a_1 + 9\,a_2$ $x = 5\ and\ y = 38 \quad \rightarrow \quad 38 = a_0 + 5\,a_1 + 25\,a_2$	$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 38 \end{bmatrix}$

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \end{bmatrix} \quad , \quad \mathbf{X} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 6 \\ 18 \\ 38 \end{bmatrix}$$

$$\boldsymbol{A = L \cdot U}$$

First, we use Gauss elimination:

row2 – (1) x row1 $\rightarrow$ row2 , $L_{21} = 1$ and,

row3 – (1) x row1 $\rightarrow$ row3 , $L_{31} = 1 \rightarrow \boldsymbol{A'} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 0 & 4 & 24 \end{bmatrix}$

row3 – (2) x row2 $\rightarrow$ row3 , $L_{32} = 2 \rightarrow \boldsymbol{U} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 0 & 0 & 8 \end{bmatrix}$ and $\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$

$$\boldsymbol{A \cdot X = B}$$

$$\boldsymbol{L \cdot (U \cdot X) = B}$$

$$\boldsymbol{Y = U \cdot X} \quad \rightarrow \quad \boldsymbol{L \cdot Y = B} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 18 \\ 38 \end{bmatrix}$$

$y_1 = 6$, $y_2 = 18 - 6 = 12$, $y_3 = 38 - 6 - 2 \cdot 12 = 8 \rightarrow \mathbf{Y} = \begin{bmatrix} 6 \\ 12 \\ 8 \end{bmatrix}$

$$\boldsymbol{U \cdot X = Y} \quad \rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 0 & 0 & 8 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 8 \end{bmatrix} \quad \rightarrow$$

$a_2 = \frac{8}{8} = 1$, $a_1 = \frac{12 - 8 \cdot (1)}{2} = 2$, $a_0 = \frac{6 - 1 - 2}{1} = 3 \quad \rightarrow \quad \begin{bmatrix} \mathbf{a_0} \\ \mathbf{a_1} \\ \mathbf{a_2} \end{bmatrix} = \begin{bmatrix} \mathbf{3} \\ \mathbf{2} \\ \mathbf{1} \end{bmatrix}$

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Question 3) [25 points] The data below is obtained from a gas sensor calibration.

v (Volts)	2.0	4.0	5.0	7.0
f (ppm)	16.0	41.0	70.0	150.0

Determine the parameters a and b so that $f(v) = a \cdot e^{b.v}$ fits the calibration data in the least-squares sense. What is the ppm concentration of a sampled gas when the sensor gives 6 volts?
Work with at least 3 decimal points. Hint: y=a+bx , $b = \frac{\sum y_i \cdot (x_i - \bar{x})}{\sum x_i \cdot (x_i - \bar{x})}$

Solution 3)

We convert $f(x)=a.e^{bx}$ to $z=c+bx$ shape, and use the curve fitting equations of a straight line:
$ln[f(x)] = ln(a.e^{bx}) \quad \rightarrow \quad ln[f(x)] = ln(a) + ln(e^{bx}) \quad \rightarrow \quad ln[f(x)] = ln(a) + b.x$.

So, the new table is:

	x_i	$f_i(x)$	$z_i=ln[f(x)]$	x_i^2	$x_i\, z_i$
	2.00	16.00	2.773	4.000	5.545
	4.00	41.00	3.714	16.000	14.854
	5.00	70.00	4.249	25.000	21.242
	7.00	150.00	5.011	49.000	35.074
Σ	12.00		15.745	94	76.716
Mean	4.50		3.936		

$$b = \frac{\sum z_i \cdot (x_i - \bar{x})}{\sum x_i \cdot (x_i - \bar{x})} = \frac{\sum x_i z_i - \bar{x} \sum z_i}{\sum x_i^2 - \bar{x} \sum x_i} = \frac{76.716 - 4.00 \cdot 15.745}{94.00 - 4.00 \cdot 12.00} \quad \rightarrow \quad \boldsymbol{b = 0.451}$$

$$\bar{z} = c + b \cdot \bar{x} \quad \rightarrow \quad c = \bar{z} - b \cdot \bar{x} = 3.936 - 0.451 \cdot 4.000 \quad \rightarrow \quad c = 1.907$$

$$c = \ln(a) \quad \rightarrow \quad a = e^c = e^{1.907} \quad \rightarrow \quad a = 6.733$$

Thus, $\mathbf{f(x) = a \cdot e^{b.x}} \quad \rightarrow \quad \mathbf{f(x) = 6.733 \cdot e^{0.451.x}}$

$$f(6) = 6.733 \cdot e^{0.451.(6.00)} = \mathbf{100.763\ ppm}$$

Error Analysis:

x_i	f_i	$\bar{f}_i$	$\varepsilon_i = f_i - \bar{f}_i$	$\%\varepsilon_i = 100 \cdot \lvert\varepsilon_i / f_i\rvert$ %
2.00	16.00	16.592	0.592	3.698%
4.00	41.00	40.888	-0.112	-0.273%
5.00	70.00	64.187	-5.813	-8.304%
7.00	150.00	158.181	8.181	5.454%

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Question 4) [25 points] Determine the closest point of the curve $y = x^2 - x + 3$ to the point $(x_o, y_o) = (0\,, 2)$ by using **Golden Bracketing** method ($R = 0.618034$).
Search x between 0.250 and 0.450. Let $\varepsilon = 0.05$ to stop the bracketing.

Solution 4) The distance equation between the curve and the point:

$$d = \sqrt{(x - x_o)^2 + (y - y_o)^2} = \sqrt{(x - x_o)^2 + (x^2 - x + 3 - y_o)^2}$$

$$(x_o, y_o) = (0,2) \quad \rightarrow \quad d = \sqrt{(x - 0)^2 + (x^2 - x + 3 - 2)^2}$$

$$\therefore \quad f(x) = \sqrt{x^2 + (x^2 - x + 1)^2} \quad to\ be\ optimized.$$

i	a_i	b_i	h	$a_i' = b_i - Rxh$	$b_i' = a_i + Rxh$	$f(a_i')$	$f(b_i')$	$x_i = (a_i + b_i)/2$	
0	0.2500	0.4500	0.2000	0.3264	0.3736	0.8457	0.8522	0.3500	
1	0.2500	0.3736	0.1236	0.2972	0.3264	0.8451	0.8457	0.3118	
2	0.2500	0.3264	0.0764	0.2792	0.2972	0.8461	0.8451	0.2882	
3	0.2792	0.3264	0.0472	0.2972	0.3084	0.8451	0.8450	0.3028	$h < \varepsilon$

$$y(x) = x^2 - x + 3 \quad \rightarrow \quad y(0.3028) = (0.3028)^2 - (0.3028) + 3 = 2.7889$$

Thus, the closest point of the function to (x_o, y_o) is **(0.3028 , 2.7889)**

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