MKT 2812 Signals and Systems

Name:__SOLUTIONS_____

ID:_____ Gr:____

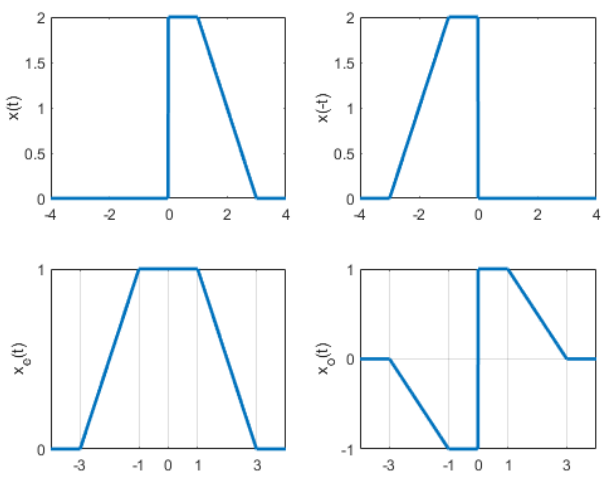
Midterm Exam

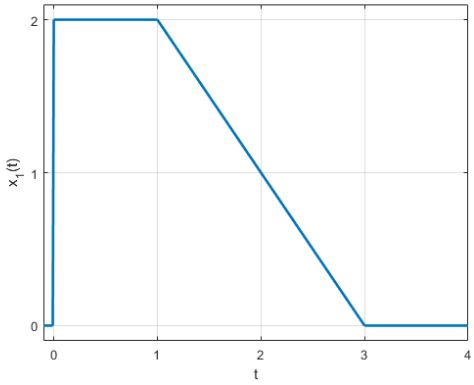
Date: Apr. 28, 2022

Total	1	2	3
100	30	40	30

1) [30 pts.] Answer the following signal questions

a) (10 pts.) Graphically compute and plot the even and odd parts of $x_1(t)$





b) (10 pts.) Let the signals be

$$x(t) = \cos\left(\frac{\pi}{3}t\right), \quad y(t) = 8\sin(2t) + 5\cos(3t), \quad z(t) = 3x(t)y(t)$$

What are the fundamental periods of signals if they are periodic?

$$T_x = \frac{2\pi}{\pi/3} = 6$$

$$T_1 = \frac{2\pi}{2} = \pi, \;\; T_2 = \frac{2\pi}{3} \rightarrow \frac{T_1}{T_2} = \frac{3}{2} \rightarrow \;\; T_y = 2T_1 = 3T_2 = 2\pi$$

$$T_z? \;\; \frac{2\pi}{6} = \frac{l}{k} \quad \nexists\, l,k \in I \;\ni\; 3l = \pi k \;\rightarrow Aperiodic$$

c) (10 pts.) Is $x(t)$ a power or energy signal? Compute both energy and power.

$$x(t) = e^{-5|t|} + \sin(t)\,u(t)$$

$$E_x = \lim_{T\to\infty}\int_{-T}^{T}\left[e^{-5|t|} + \sin(t)u(t)\right]^2 dt = \lim_{T\to\infty}\int_{0}^{T}\left[2e^{-10t} + 2e^{-5t}\sin(t) + \sin^2 t\right]dt$$

Using the integral equations at the end

$$E_x = \lim_{T\to\infty}\left[\left(-\frac{2}{10}e^{-10t}\right) + \frac{2}{25+1}e^{-5t}(-5\sin t - \cos t) + \left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\right]_{t=0}^{T} = \infty$$

$$P_x = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[e^{-5|t|} + \sin(t)u(t)\right]^2 dt = \lim_{T\to\infty}\frac{1}{2T}\int_{0}^{T}\left[2e^{-10t} + 2e^{-5t}\sin(t) + \sin^2 t\right]dt$$

$$P_x = \lim_{T\to\infty}\frac{1}{2T}\left[\left(-\frac{1}{5}e^{-10t}\right) + \frac{1}{13}e^{-5t}(-5\sin t - \cos t) + \left(\frac{1}{2}t - \frac{1}{4}\sin 2t\right)\right]_{t=0}^{T}$$

$$P_x = \lim_{T\to\infty}\frac{1}{2T}\left\{\left(-\frac{1}{5}e^{-10T}\right) + \frac{1}{13}e^{-5T}(-5\sin T - \cos T) + \left(\frac{1}{2}T - \frac{1}{4}\sin 2T\right) - \left[-\frac{1}{5} - \frac{1}{13}\right]\right\} = \frac{1}{4}$$

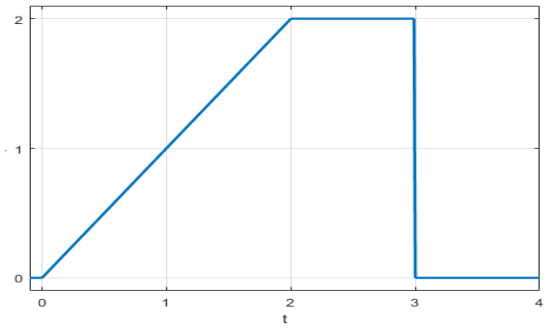
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2) [40 pts.]

a) (20 pts.) Analyze the linearity, time-invariance, BIBO stability and causality of the system with input $x(t)$ output $y(t)$. Find the impulse response $h(t)$

$$y(t) = 2\int_{-\infty}^{t-1} \sin(3\tau)\, x(\tau) d\tau$$

b) (20 pts.) For another LTI system with the impulse response of



find its step response (to the input $x(t) = u(t)$) using the convolution integral

a) Linear?

$$y(t) = 2\int_{-\infty}^{t-1} \sin(3\tau)\,[x_1(\tau) + ax_2(\tau)] d\tau = 2\int_{-\infty}^{t-1} \sin(3\tau)\, x_1(\tau) d\tau + a \cdot 2\int_{-\infty}^{t-1} \sin(3\tau)\, x_2(\tau) d\tau = y_1 + ay_2(t)$$

So, it is linear.

Time-invariant?

Is the delayed output

$$y(t-T) = 2\int_{-\infty}^{t-T-1} \sin(3\tau)\, x(\tau) d\tau \quad (*)$$

Equals to the delayed input response

$$2\int_{-\infty}^{t-1} \sin(3\tau)\, x(\tau - T) d\tau$$

For clarity, define a change of variable $\tau - T = \lambda$ when $\tau = t-1$ then $\lambda = t - 1 - T$ and $d\tau = d\lambda$, thus

$$2\int_{-\infty}^{t-1-T} \sin\big(3(\lambda + T)\big)\, x(\lambda) d\tau$$

$\sin(3t)$ is not equal to $\sin\big(3(t+T)\big)$ $\forall T \neq 2\pi k/3$ $for\ k = 0 \pm 1 \pm 2$ ....

So, time varying. Or, pick $x(t) = u(t)$ and $T = 1$ and show this counter example to prove it is time varying.

BIBO Stable?

Let $|x(t)| < B_1$ $\forall t$, then

$$y(t) \leq |2\int_{-\infty}^{t-1} \sin(3\tau)\, B_1 d\tau| = |\frac{2B_1}{3}\cos(3\tau)|_{\infty}^{t-1}| < \frac{4B_1}{3} < \infty$$

So, BIBO stable.

Causal?

For $x = 0 \rightarrow$ $y = 0$ and the output at time t depends only on the past inputs, so, causal.

b) $h(t) = r(t) - r(t-2) - 2u(t-3)$

$$y(t) = [h * u](t) = \int_{-\infty}^{\infty} u(t-\tau)[r(\tau) - r(\tau - 2) - 2u(\tau - 3)] d\tau$$

$$y(t) = u(t)\int_0^t \tau d\tau - u(t-2)\int_2^t (\tau - 2) d\tau - 2u(t-3)\int_3^t d\tau$$

$u(t)$s must be added to make sure the integral upper and lower bounds in order.

$$y(t) = \frac{t^2}{2}u(t) - \frac{(t-2)^2}{2}u(t-2) - 2(t-3)u(t-3)$$

$$y(t) = \frac{t}{2}r(t) - \frac{t-2}{2}r(t-2) - 2r(t-3)$$

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3) [30 pts.] a) For a system with impulse response

$$H(s) = \frac{5s^2 + 4s + 4}{s(s^2 + 2s + 2)} \quad ROC: -1 < \sigma < 0$$

i. Find the poles and zeros
ii. Determine if it is causal, anti-causal or non-causal system?
iii. Draw its inverse Laplace transformation $h(t)$

i. Poles $0, -1 \pm j$, Zeros , $-0.4 \pm j0.8$
ii. Non-causal
iii. PFE

$$H(s) = \frac{c_o}{s} + \frac{As + B}{s^2 + 2s + 2}$$

$$c_o = sH(s)|_{s=0} = \frac{4}{2} = 2$$

$$\frac{As + B}{s^2 + 2s + 2} = H - \frac{2}{s} = \frac{5s^2 + 4s + 4}{s(s^2 + 2s + 2)} - \frac{2(s^2 + 2s + 2)}{s(s^2 + 2s + 2)} = \frac{3s}{s^2 + 2s + 2}$$

So, $A = 3, B = 0$. Thus, the PFE

$$H(s) = \frac{2}{s} + \frac{3(s+1) - 3}{(s+1)^2 + 1}$$

$R_1 = \{\sigma > -1\}$ is the ROC of the causal part and can't include any (causal) poles. So, poles $-1 \pm j$ are causal pole pairs.

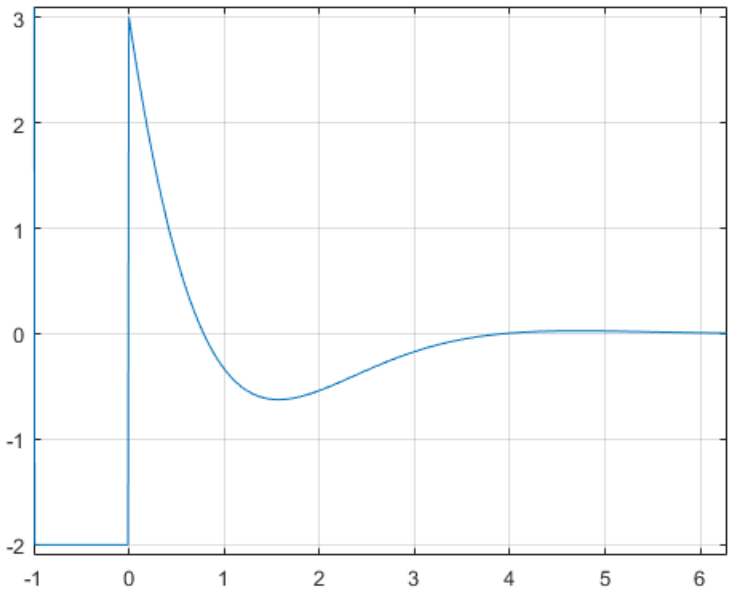
$R_2 = \{\sigma < 0\}$ is the ROC of the anti-causal part and can't include any (a.c.) poles. So, the pole 0 are a.c. pole.

$H_c(s) = \frac{3(s+1)-3}{(s+1)^2+1}$ → Inverse Laplace transform thru the Table $h_c(t) = 3e^{-t}[\cos t - \sin t]u(t)$

Only $G(s) = H_{ac}(-s) = -\frac{2}{s}$ Inverse transform may be obtained thru the Table as $g(t) = -2u(t)$. Thus,

$h_{ac}(t) = g(-t) = -2u(-t)$. Therefore,

$$h(t) = h_c(t) + h_{ac}(t) = 3e^{-t}[\cos t - \sin t]u(t) - 2u(-t)$$



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Useful Equations

Basic Signals

$$\frac{dr}{dt} = u(t).$$

$$\frac{du}{dt} = \delta(t).$$

$$u(t) = \int_{-\infty}^{t} \delta(a)da \quad \Rightarrow \quad \int_{t_1}^{t_2} \delta(a-1)da = u(t_2-1) - u(t_1-1)\,.$$

$$r(t) = tu(t) = \int_{-\infty}^{t} u(a)da \quad \Rightarrow \quad \int_{t_1}^{t_2} u(a-1)da = r(t_2-1) - r(t_1-1).$$

Laplace transform table

$f(t)$	$F(s)$
$\delta(l)$	1
$u(t)$	$\frac{1}{s}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin \omega t u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t u(t)$	$\frac{s}{s^2+\omega^2}$

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt, \qquad s \in ROC.$$

Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{j\Omega_o kt} \text{ with } X_k = \frac{1}{T_o}\int_{t_o}^{t_o+T_o} x(t)e^{-jk\Omega_o t}\,dt \text{ or } X_k = \frac{1}{T_0}\, X_1(s)|_{s=j\Omega_o k}$$

Fourier Transform

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}\,dt \quad \text{or} \quad X(\Omega) = X(s)|_{s=j\Omega} \quad \text{if} \quad s = j\Omega \quad \text{exists in ROC}$$

Fourier Transform Pairs

$\delta(t)$, $\delta(t-\tau)$	1, $e^{-j\Omega\tau}$
$u(t)$, $u(-t)$	$\frac{1}{j\Omega} + \pi\delta(\Omega)$, $\frac{-1}{j\Omega} + \pi\delta(\Omega)$
$\text{sgn}(t) = 2[u(t) - 0.5]$	$\frac{2}{j\Omega}$
A, $Ae^{-at}u(t)$, $a > 0$	$2\pi A\delta(\Omega)$, $\frac{A}{j\Omega + a}$
$Ate^{-at}u(t)$, $a > 0$	$\frac{A}{(j\Omega + a)^2}$
$e^{-a\lvert t\rvert}$, $a > 0$	$\frac{2a}{a^2 + \Omega^2}$
$\cos(\Omega_0 t)$, $-\infty < t < \infty$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$
$\sin(\Omega_0 t)$, $-\infty < t < \infty$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$
$p(t) = A[u(t+\tau) - u(t-\tau)]$,	$2A\tau\frac{\sin(\Omega\tau)}{\Omega\tau}$

Basic Properties of One-sided Laplace Transforms

Causal functions and constants	$\alpha f(t)$, $\beta g(t)$	$\alpha F(s)$, $\beta G(s)$
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Multiplication by t	$t\,f(t)$	$-\frac{dF(s)}{ds}$
Derivative	$\frac{df(t)}{dt}$	$sF(s) - f(0-)$
Second derivative	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0-) - f^{(1)}(0)$
Integral	$\int_{0-}^{t} f(t')dt'$	$\frac{F(s)}{s}$
Expansion/contraction	$f(\alpha t)$, $\alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}F\left(\frac{s}{\alpha}\right)$
Initial value	$f(0-) = \lim_{s\to\infty} sF(s)$	

Table 5.1 Basic properties of the Fourier transform

Expansion/contraction	$x(\alpha t)$, $\alpha \neq 0$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{\Omega}{\alpha}\right)$
Reflection	$x(-t)$	$X(-\Omega)$
Parseval's	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\Omega)\rvert^2 d\Omega$
Duality	$X(t)$	$2\pi x(-\Omega)$
Differentiation	$\frac{d^n x(t)}{dt^n}$, $n \geq 1$	$(j\Omega)^n X(\Omega)$
Integration	$\int_{-\infty}^{t} x(t')dt'$	$\frac{X(\Omega)}{j\Omega} + \pi X(0)\delta(\Omega)$
Shifting	$x(t-\alpha)$, $e^{j\Omega_0 t}x(t)$	$e^{-j\alpha\Omega}X(\Omega)$, $X(\Omega - \Omega_0)$
Modulation	$x(t)\cos(\Omega_c t)$	$0.5[X(\Omega - \Omega_c) + X(\Omega + \Omega_c)]$
Periodic	$x(t) = \sum_k X_k e^{jk\Omega_0 t}$	$X(\Omega) = \sum_k 2\pi X_k \delta(\Omega - k\Omega_0)$
Symmetry	$x(t)$ real	$\lvert X(\Omega)\rvert = \lvert X(-\Omega)\rvert$, $\angle X(\Omega) = -\angle X(-\Omega)$
Convolution	$z(t) = [x * y](t)$	$Z(\Omega) = X(\Omega)Y(\Omega)$

Some Integrals:

$$\int \sin^2 wt\,dt = \frac{1}{2}t - \frac{1}{4w}\sin(2wt)$$

$$\int \cos^2 wt\,dt = \frac{1}{2}t + \frac{1}{4w}\sin(2wt)$$

$$\int e^{at}\sin wt\,dt = \frac{1}{a^2+w^2}e^{at}[a\sin wt - w\cos wt]$$

$$\int e^{at}\cos wt\,dt = \frac{1}{a^2+w^2}e^{at}[w\sin wt + a\cos wt]$$

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