Name: ____SOLUTIONS_____
ID: _____

MKT2812 Final Exam

Spring 2024

May 30, 2024

Answer ONLY 100 point questions out of 120 possible

1) [5 pts.] Assume that EEG signal is showing noticeable power between 0-100 kHz band. What is the minimum sampling frequency of EEG signal if you are to process information between 0-2 kHz range? How would you connect ADC to the sensor for such measurements?

$f_{s_{min}} =$ 4kHz

Anti-aliasing LPF with a cutoff at 2 kHz must be applied before the sampling.

2) [5 pts.] Find the period and fundamental frequency of DT signal x[n], if it is periodic.

$$x[n] = \sin\left(\frac{2\pi n}{5}\right) e^{j3\pi n} + \cos(3\pi n)$$

$N_x = 10$

$\omega_o = 2\pi/10$

$$\omega_1 = \frac{2\pi}{5} = \frac{2\pi}{N_1} m_1 \rightarrow N_1 = 5 \ \ (with\ m_1 = 1)$$

$$\omega_2 = 3\pi = \frac{2\pi}{N_2} m_2 \rightarrow N_2 = 2\ (with\ m_2 = 3)$$

$$\frac{N_1}{N_2} = \frac{m}{l} = \frac{5}{2} \ \rightarrow \ N_o = 2N_1 = 5N_2 = 10\ samples$$

NOTE: N always MUST be an integer!!!!

3) [5 pts.] What is the odd component of $x[n]$?

$x_o[n] = (-1)^n \sin\left(\frac{2\pi n}{5}\right)$

$$x[-n] = -\sin\left(\frac{2\pi n}{5}\right) e^{-j3\pi n} + \cos(3\pi n)$$

$$x_o[n] = 0.5(x[n] - x[-n])$$

$$x_o[n] = 0.5\left[\sin\left(\frac{2\pi n}{5}\right) e^{j3\pi n} + \cos(3\pi n) + \sin\left(\frac{2\pi n}{5}\right) e^{-j3\pi n} - \cos(3\pi n)\right]$$

$$x_o[n] = \sin\left(\frac{2\pi n}{5}\right) 0.5[e^{j3\pi n} + e^{-j3\pi n}] = \sin\left(\frac{2\pi n}{5}\right)\cos(3\pi n) = (-1)^n \sin\left(\frac{2\pi n}{5}\right)$$

tonozdayim.com/@Zedeleyici31

4) [5 pts] Show if $z[n]$ is an energy signal or not?

$$z[n] = 0.4^n u[n] + 2^n u[-n]$$

$E_z = 4.53 < \infty$

$$|z[n]|^2 = 0.4^{2n} u[n] + 2^{2n} u[-n] + 2 \cdot 0.8^n u[n] u[-n]$$

$$E_z = \lim_{N\to\infty} \sum_{n=-N}^{N} (0.4^{2n} u[n] + 2^{2n} u[-n] + 2 \cdot 0.8^n u[n] u[-n])$$

$$E_z = \lim_{N\to\infty} \left\{ \sum_{n=0}^{N} 0.16^n + \sum_{m=0}^{N} 4^{-m} + 2 \right\} = \lim_{N\to\infty} \left\{ \sum_{n=0}^{N} 0.16^n + \sum_{m=0}^{N} \frac{1}{4}^m + 2 \right\}$$

$$E_z = \frac{1}{1-0.16} + \frac{1}{1-0.25} + 2 = 4.53 < \infty$$

tonozdayim.com/@Zedeleyici31

5) [20 pts.] Let DT system be

$$y[n] = \sum_{k=-\infty}^{n+1} 0.5^{n-k} x[k]$$

Is it linear, time-invariant, causal, BIBO stable?

Linear?
Time-invariant?
Causal?
BIBO stable?

$$y[n] = \sum_{k=-\infty}^{n+1} 0.5^{n-k}(x_1[k] + ax_2[k]) = \sum_{k=-\infty}^{n+1} 0.5^{n-k} x_1[k] + a \sum_{k=-\infty}^{n+1} 0.5^{n-k} x_2[k]$$

$$= y1 + ay_2$$

So **linear**.

N Delayed output:

$$\sum_{k=-\infty}^{n-N+1} 0.5^{n-N-k} x[k]$$

N Delayed-input response:

$$\sum_{k=-\infty}^{n+1} 0.5^{n-k} x[k-N]$$

Make them look alike by a change of variable as $k - N = m$ so

$$\sum_{m=-\infty}^{n-N+1} 0.5^{n-N-m} x[m]$$

N Delayed output is the same N Delayed-input response. **Time Invariant.**

Output at time n depends on input at time n+1 (Future) **Non-causal**

Let $|x[k]| < B_x \ \ for\ all\ k$. Then

$$|y[n]| < B_x \sum_{k=-\infty}^{n+1} 0.5^{n-k}$$

Define $n - k = m$

$$|y[n]| < B_x \sum_{m=-1}^{\infty} 0.5^m = B_x \left(0.5^{-1} + \sum_{m=0}^{\infty} 0.5^m \right) = B_x \left(2 + \frac{1}{1-0.5} \right) = 4B_x = B_y < \infty$$

BIBO stable

tonozdayim.com/@Zedeleyici31

6) [10 pts.] Impulse response?

$h[n] = 0.5^n u[n+1]$

$$h[n] = \sum_{k=-\infty}^{n+1} 0.5^{n-k}\delta[k] = 0.5^n \sum_{k=-\infty}^{n+1} \delta[k] = 0.5^n u[n+1]$$

tonozdayim.com/@Zedeleyici31

7) [15pts] Find the Fourier series coefficients of $x(t)$, with the period of 4 sec. and the one-period signal

$$x_1(t) = r(t+1) - 2r(t) + r(t-1)$$

$$X_k = 2\frac{1-\cos\left(\frac{\pi}{2}k\right)}{\pi^2 k^2} \quad \forall k \neq 0,$$

$$X_o = 1$$

The Laplace transform

$$X_1(s) = \frac{(e^s - 2 + e^{-s})}{s^2} \qquad ROC = whole\ s - plane$$

ROC includes all $j\Omega_o k$'s,

$$X_k = \frac{1}{T_o} X_1(s)|_{s=j\Omega_o k} = \frac{1}{4}\frac{\left(e^{j\frac{\pi}{2}k} - 2 + e^{-j\frac{\pi}{2}k}\right)}{-\left(\frac{\pi}{2}\right)^2 k^2} = \frac{\left(2 - e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)}{\pi^2 k^2}, \qquad \forall k \neq 0$$

$$X_o = 1$$

You may further simplify it (the rest is optional)

$$X_k = 2\frac{1-\cos\left(\frac{\pi}{2}k\right)}{\pi^2 k^2}$$

Or

$$X_k = \frac{-1}{\pi^2 k^2}\left(e^{j\frac{\pi}{4}k} - e^{-j\frac{\pi}{4}k}\right)^2 = \frac{4}{\pi^2 k^2}\left(\frac{e^{j\frac{\pi}{4}k} - e^{-j\frac{\pi}{4}k}}{2j}\right)^2 = \frac{1}{4}\left(\frac{1}{\frac{\pi}{4}k}\right)^2 \sin\left(\frac{\pi}{4}k\right)^2$$

$$X_k = \frac{1}{4} sinc\left(\frac{\pi}{4}k\right)^2$$

And

$$X_o = 1$$

Must be specified.

8) [15pts.] Find the Fourier transform of

$$x(t) = e^{-5t}\sin(4t)\,u(t)$$

$$X(\Omega) = \frac{4}{41 - \Omega^2 + j10\Omega}$$

The Laplace transform

$$X(s) = \frac{4}{(s+5)^2 + 16} \qquad ROC = \{\sigma > -5\}$$

Sine ROC includes $j\Omega$, laplace transform may be used for the fourier transform as

$$X(\Omega) = \frac{4}{(j\Omega + 5)^2 + 16} = \frac{4}{41 - \Omega^2 + j10\Omega}$$

tonozdayim.com/@Zedeleyici31

9) [20 pts.] A bank is giving housing credit of L lira with monthly interest of $\alpha\%$. Find the monthly payment you need to make to pay the total loan in 10 years.
Hints: First write the difference equations $y[n]$ being your total debt at the n^{th} month. Let $x[n] = X$ be your constant monthly payment. Recursively, write the total debt starting from month zero to the 120th month (10 years) and solve for the unknown X symbolically

$$X = \frac{r^{121} - r^{120}}{r^{120} - 1} L \text{ with } r = 1 + a\%$$

First write the difference equations $\mathrm{y[n]}$ being your total debt at the $\mathrm{n^{th}}$ month. Let $\mathrm{x[n] = X}$ be your constant monthly payment at the nth month. Thus

$$y[n] = (1 + a\%)y[n-1] - x[n]$$

With $\mathrm{y[0] = y_o = 2\ Million}$

Let $(1 + a\%) = r$. Recursively, write the total debt

$$y[1] = ry_o - X$$
$$y[2] = ry[1] - X = r(ry_o - X) - X$$
$$y[2] = r^2 y_o - (r+1)X$$
$$\vdots$$
$$y[N] = r^N y_o - \sum_{k=0}^{N-1} r^k X$$

Or geometric sum rewritten as

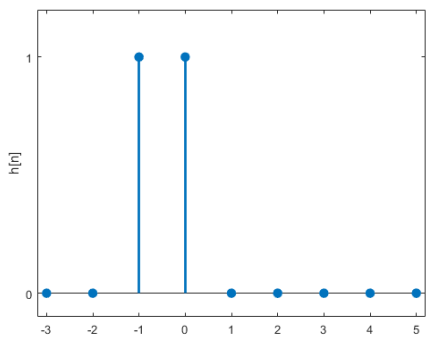
$$y[N] = r^N y_o - \frac{1 - r^N}{1 - r} X$$

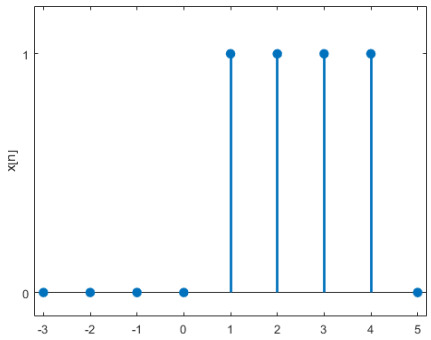
For 10 year plan N=120 the total debt must be paid off, the balance, $y[120] = 0$,

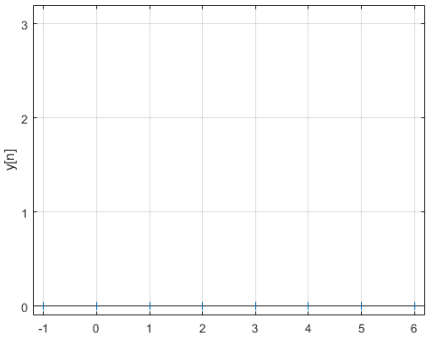
$$X = \frac{r^{120}(r-1)}{r^{120} - 1} L$$

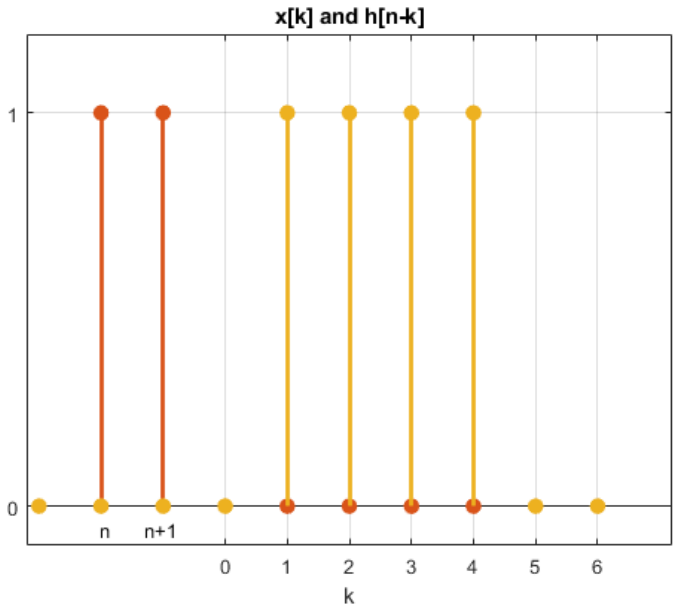
tonozdayim.com/@Zedeleyici31

10) [20 pts.] For an LTI DT system represented by h[n] below, find its response to the input x[n]









When $n+1<1$ then $y[n]=0$ since they don't overlap as seen from Fig. above. Hence,

$$n<0 \rightarrow y[n]=0$$
$$n+1=1 \rightarrow y[0]=1$$
$$n+1=2 \rightarrow y[1]=1+1=2$$
$$n+1=3 \rightarrow y[2]=1+1=2$$
$$n+1=4 \rightarrow y[3]=1+1=2$$
$$n+1=5 \rightarrow y[4]=1$$
$$n>4 \rightarrow y[n]=0$$

tonozdayim.com/@Zedeleyici31