MKT 2812 Signals and Systems

Name:_____

ID:_____ Gr:____

Final Exam

Date: Jun. 13, 2022

| Total | 1 | 2 | 3 | 4 | 5 |
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| | | | | | |

1) [25 pts.] Let the DT system be

$$y[n] = \cos(20\pi n - \pi/3)x[n]$$

*i.* (5 pts.) Time-invariant? Prove it.

a) Yes b) No

$$y[n-N] = \cos(20\pi n - 2\pi 10N - \pi/3)x[n-N]$$

$$Delayed\ response\ = \cos(20\pi n - \pi/3)x[n-N]$$

Both equal if $\cos(20\pi n - 2\pi 10N - \pi/3) = \cos(20\pi n - \pi/3) \quad \forall N \in \mathbb{I}$. YES it is Time Invariant.

2) [20 pts.] Find the Fourier series representation of periodic signal $x(t)$, with period of 4 sec. and one period $x_1(t) = u(t-1) - u(t-3)$.

a)

b) (13 pts.) What is the amplitude of the second harmonic ($A$ in $A\cos(2\Omega_o t)$ term)?

c)

$$T_o = 4\ \Omega_o = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$X_k = \frac{1}{4}\int_1^3 e^{-j\frac{\pi}{2}kt}dt = \frac{1}{\pi k}\frac{\left(e^{-j\frac{\pi}{2}k} - e^{-j3\frac{\pi}{2}k}\right)}{2j} = \frac{1}{\pi k}e^{-j\pi k}\frac{\left(e^{j\frac{\pi}{2}k} - e^{-j\frac{\pi}{2}k}\right)}{2j} = \frac{1}{\pi k}(-1)^k \sin\frac{\pi}{2}k$$

$$X_2 = 0 = X_{-2} \rightarrow A_2 = 0$$

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3) [25 pts.] Let system be defined by

$$y(t) = 5\int_{-\infty}^{t} e^{-2(t-\tau)}x(\tau-1)d\tau$$

(5 pts.) Describe methods to find CT LTI system response to an arbitrary input?

(20 pts.) Find the system response to the input $x(t) = r(t) - 2r(t-1) + r(t-2)$ using Laplace transform

a) Find the impulse response $h(t) = S[\delta(t)]$ then apply the convolution integral in time domain

$$y(t) = [h * x](t)$$

or Laplace transform input $X(s)$ and impulse response $H(s)$ multiplication's inverse laplace

$$y(t) = L^{-1}\{H(s)X(s)\}$$

Direct approach $y(t) = S[x(t)]$

b) $h(t) = 5\int_{-\infty}^{t} e^{-2(t-\tau)}\delta(\tau-1)d\tau = 5e^{-2(t-1)}\int_{-\infty}^{t}\delta(\tau-1)d\tau = 5e^{-2(t-1)}u(t-1)$

$$H(s) = \frac{5}{s+2}e^{-s}$$

$$X(s) = \frac{1-2e^{-s}+2e^{-2s}}{s^2}$$

$$Y(s) = HX = \frac{5}{s^2(s+2)}(e^{-s}-2e^{-2s}+2e^{-3s})$$

Define the core term (without the delays)

$$Z(s) = \frac{5}{s^2(s+2)}$$

PFE for simplified terms

$$Z(s) = \frac{5}{s^2(s+2)} = \frac{c_o}{s+2} + \frac{c_1}{s^2} + \frac{c_2}{s}$$

Find the coefficients

$$c_o = (s+2)Z|_{s=-2} = \frac{5}{(-2)^2} = \frac{5}{4}$$

$$c_1 = s^2Z|_{s=0} = \frac{5}{(2)^2} = \frac{5}{4}$$

$$c_2 = \frac{d}{ds}\{s^2Z\}\,|_{s=0} = \frac{-5}{(2)^2} = -\frac{5}{4}$$

$$z(t) = \frac{5}{4}\{-1+t+e^{-2t}\}\,u(t)$$

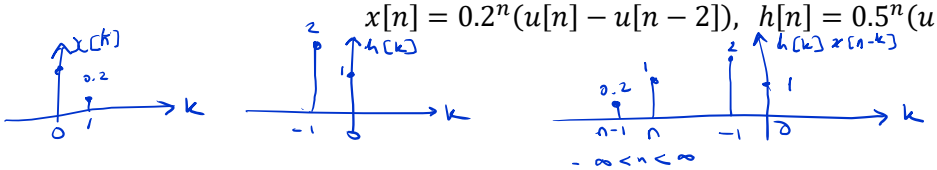
Using the LTI time shift prop.

$$y(t) = \frac{5}{4}\{-z(t-1)+z(t-2)+z(t-3)\}$$

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4) [10 pts] Find the convolution sum of $(x \otimes h)[n]$ where

$$x[n] = 0.2^n(u[n] - u[n-2]), \quad h[n] = 0.5^n(u[n+1] - u[n-1])$$




$n < -1$ $\quad y[n] = 0$

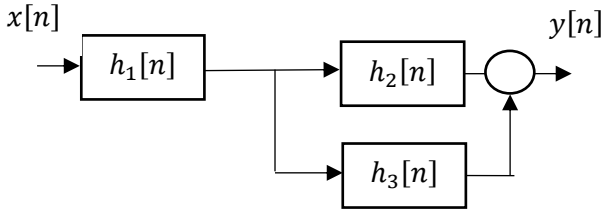
$n = -1$ $\quad y[-1] = 2$

$n = 0$ $\quad y[0] = 2 \times 0.2 + 1 \cdot 1 = 1.4$

$n = 1$ $\quad y[1] = 1 \times 0.2 = 0.2$

$n > 1$ $\quad y[n] = 0$

$$\Rightarrow \quad y[n] = 2\delta[n+1] + 1.4\,\delta[n] + 0.2\,\delta[n-1]$$




$$h_1[n] = 2(u[n] - u[n-3])$$
$$h_2[n] = 0.4^n u[n]$$
$$h_3[n] = \delta[n] - \delta[n-1]$$

5) [20 pts.] What is the equivalent impulse response of the overall system shown above?

In time domain:

$$h[n] = h_1[n] * (h_2[n] + h_3[n]) = (h_1 * h_2)[n] + (h_1 * h_3)[n]$$

$a)$ Note that we have general terms of $u[n] * r^n u[n-a]$ for $r \in \mathbb{R}, a \in \mathbb{I}$, the convolution sum

$$u[n] * r^n u[n-a] = \sum_{k=-\infty}^{\infty} u[n-k] r^k u[k-a] = u[n-a] \sum_{k=a}^{n} r^k = u[n-a] \left\{ \frac{(1-r^{n+1})}{1-r} - \frac{(1-r^a)}{1-r} \right\}$$

$$u[n] * r^n u[n-a] = \left\{ \frac{r^a - r^{n+1}}{1-r} \right\} u[n-a] = \frac{r^a}{1-r} \{1 - r^{n+1-a}\} u[n-a] \qquad (1)$$

$b)$ $u[n+a] * \delta[n+b]$ for $a, b \in \mathbb{I}$, the convolution sum

$$u[n+a] * \delta[n-b] = \sum_{k=-\infty}^{\infty} u[n-k+a]\delta[k-b] = \sum_{k=-\infty}^{n+a} \delta[k-b] = \sum_{m=-\infty}^{n+a-b} \delta[m] = u[n+a-b] \qquad (2)$$

These general results may be reused for various problems. Apply this for the final question where

$$(h_1 * h_2)[n] = 2\big(u[n] * (0.4^n u[n]) - u[n-3] * (0.4^n u[n])\big)$$

$$u[n] * (0.4^n u[n]) = \frac{1}{0.6}(1 - 0.4^{n+1}) u[n] \qquad (r = 0.4, a = 0)$$

$$u[n-3] * (0.4^n u[n]) = \frac{0.4^{-3}}{0.6}(1 - 0.4^{n-2}) u[n-3] \quad (r = 0.4, a = 3)$$

$$\boxed{(h_1 * h_2)[n] = \frac{10}{3}\big((1 - 0.4^{n+1}) u[n] - 0.4^{-3}(1 - 0.4^{n-2}) u[n-3]\big)} \qquad (3)$$

The second term

$$(h_1 * h_3)[n] = 2(u[n] - u[n-3]) * (\delta[n] - \delta[n-1])$$

May be computed using the general template (2) as

$$u[n] * \delta[n] = u[n] \qquad (a = 0, b = 0)$$
$$u[n] * \delta[n-1] = u[n-1] \qquad (a = 0, b = 1)$$
$$u[n-3] * \delta[n] = u[n-3] \qquad (a = -3, b = 0)$$
$$u[n-3] * \delta[n-1] = u[n-4] \quad (a = -3, b = 1)$$

$$\boxed{(h_1 * h_3)[n] = 2(u[n] + u[n-1] - u[n-3] + u[n-4]) = 2(\delta[n] - \delta[n-3])} \qquad (4)$$

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In z (frequency) domain, the solution would have been shorter

$$H(z) = H_1(z)[H_2(z) + H_3(z)]$$

$$h_1[n] = 2(u[n] - u[n-3]) = 2\delta[n] + 2\delta[n-1] + 2\delta[n-2] \rightarrow H_1(z) = 2(1 + z^{-1} + z^{-2})$$

$$h_2[n] = 0.4^n u[n] \rightarrow H_2(z) = \frac{1}{1 - 0.4z^{-1}}$$

$$h_3[n] = \delta[n] - \delta[n-1] \rightarrow H_3(z) = 1 - z^{-1}$$

$$H(z) = H_1[H_2 + H_3] = 2(1 + z^{-1} + z^{-2})\left[\frac{1}{1 - 0.4z^{-1}} + 1 - z^{-1}\right]$$

$$H(z) = 2\left(\frac{1}{1 - 0.4z^{-1}} + z^{-1}\frac{1}{1 - 0.4z^{-1}} + z^{-2}\frac{1}{1 - 0.4z^{-1}} + 1 - z^{-3}\right)$$

Inverse z-xform

$$h[n] = 2(h_2[n] + h_2[n-1] + h_2[n-2] + \delta[n] - \delta[n-3])$$

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