2024-2025 FALL / MAT1071 MATHEMATICS 1
2nd MIDTERM

(1) For a differentiable function $g:\mathbb{R}\to\mathbb{R}$ where $g(0)=0$ and $a,b\in\mathbb{R}$, let

$$f:\mathbb{R}\to\mathbb{R},\quad f(x)=\begin{cases}\dfrac{g(x)}{x}, & x<0\\ a+g(bx), & x\geq 0.\end{cases}$$

If the equation of the line tangent to the curve $y=f(x)$ at the point $P(0,a)$ is $y=a+x$, find the value of $a\cdot b$.

Start with the limit of $f(x)$ at $x=0$.

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}[a+g(bx)]=a+g(0)=a.$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{g(x)}{x}\underset{(\frac{0}{0})}{\overset{L}{=}}\lim_{x\to 0^-}\frac{g'(x)}{1}=g'(0)$$

$$\Big\}\ g'(0)=a.$$

Tangent line: $y=a+x \Rightarrow m_T=1 \ (y'=1) \Rightarrow f'(0)=1$

$f'(x)=(a+g(bx))'=b\cdot g'(bx)$ (We know that the derivative exists)

$f'(0)=b\cdot g'(0)=1 \Rightarrow a\cdot b=1$

(2) For the function $f:(0,\frac{\pi}{2})\to\mathbb{R}$, $f(x)=\operatorname{arccot}\sqrt{\frac{1+\cos x}{1-\cos x}}$, what is the value of $f(1)+f'(1)=?$

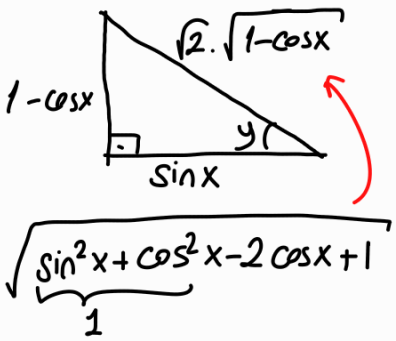
$$y=\operatorname{arccot}\sqrt{\frac{1+\cos x}{1-\cos x}} \Rightarrow \cot y=\sqrt{\frac{1+\cos x}{1-\cos x}}=\sqrt{\frac{1-\cos^2 x}{(1-\cos x)^2}}=\frac{\sin x}{1-\cos x}$$

($\sin^2 x$; $(1-\cos x)$)

(Easier to evaluate)

$$\sin y=\frac{1-\cos x}{\sqrt{2}\cdot\sqrt{1-\cos x}}=\frac{\sqrt{1-\cos x}}{\sqrt{2}}$$

(Take derivative)



$$y'\cdot\cos y=\frac{1}{\sqrt{2}}\cdot\frac{\sin x}{2\sqrt{1-\cos x}} \Rightarrow y'=\frac{1}{2} \Rightarrow y=\frac{x}{2}+c$$

($\cos y=\frac{\sin x}{\sqrt{2}\cdot\sqrt{1-\cos x}}$)

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$f'(x)=\frac{1}{2} \Rightarrow f'(1)=\frac{1}{2}$. Now find c. (Pick an $x\in(0,\frac{\pi}{2})$)

Let $x=\frac{\pi}{3}$. $\sin y=\frac{\sqrt{1-\cos x}}{\sqrt{2}} \Rightarrow \sin\left(y\left(\frac{\pi}{3}\right)\right)=\frac{\sqrt{1-\cos\frac{\pi}{3}}}{\sqrt{2}}=\frac{1}{2} \Rightarrow y=\frac{\pi}{6}$

$y=\frac{x}{2}+c \Rightarrow \frac{\pi}{6}=\frac{\pi}{3.2}+c \Rightarrow c=0 \Rightarrow f(x)=\frac{x}{2} \Rightarrow f(1)=\frac{1}{2}$

$\Rightarrow f(1)+f'(1)=\frac{1}{2}+\frac{1}{2}=$ **1**

(3) If f is a continuously differentiable even function where $f(1)=2$ and $\int_0^1 \arctan x \cdot f'(x)\,dx=\frac{\pi}{2}-1$ then what is the value of $\int_{-1}^{1}\frac{f(x)}{1+x^2}\,dx=?$

$\int_{-1}^{1}\frac{f(x)\,dx}{1+x^2}=2\cdot\underbrace{\int_0^1\frac{f(x)}{1+x^2}\,dx}_{A}=2.A$ (f is even, $1+x^2$ is even)

$\int_0^1 \arctan x\cdot f'(x)\,dx=\frac{\pi}{2}-1$

$\arctan x=u \quad f'(x)\,dx=dv$

$\frac{1}{1+x^2}dx=du \quad f(x)=v$

$\Rightarrow \underbrace{\arctan x\cdot f(x)\Big|_0^1}_{\arctan 1\cdot f(1)-0}-\underbrace{\int_0^1\frac{f(x)}{1+x^2}\,dx}_{A}=\frac{\pi}{2}-1 \Rightarrow 2\cdot\frac{\pi}{4}-A=\frac{\pi}{2}-1 \Rightarrow A=1$

$\Rightarrow 2A=$ **2**

(4) Which of the following is the result of the integral $\int_{-1}^{1}\frac{x^2.\sin x}{1+x^4}\,dx=?$

$f(x)=\frac{x^2.\sin x}{1+x^4} \Rightarrow f(-x)=\frac{(-x)^2.\sin(-x)}{1+(-x)^4}=\frac{-x^2.\sin x}{1+x^4}=-f(x)$

$\Rightarrow f(x)$ is odd. $\int_{-1}^{1}\frac{x^2.\sin x}{1+x^4}\,dx=$ **0**

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(5) If $f: D(f) \to \mathbb{R}$ is a continuous and even function satisfying the following equation $\int_1^{\cos x} f(t)\,dt = 1 - \sin x + \int_0^{\pi+x} f(\cos t)\,dt$

then which of the following is equal to $f(x)$?

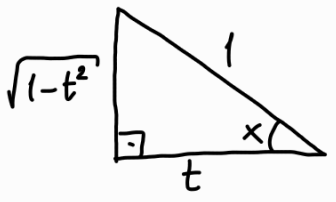
A) $\frac{1}{1+x}$ B) $\sqrt{1+x^2}$ C) $\frac{x}{1+\sqrt{1-x^2}}$ D) x E) 0

Take the derivative of both sides.

$-\sin x \cdot f(\cos x) = -\cos x + f(\underbrace{\cos(\pi+x)}_{-\cos x})$ $\Rightarrow$ $f(\cos x) = \frac{\cos x}{1+\sin x}$

$f(\cos x)$ (f is even)

Let $x = \arccos t$ $(\cos x = t)$



$x = \arccos t = \arcsin\sqrt{1-t^2}$

$$f(t) = \frac{t}{1+\sin(\arccos t)} = \frac{t}{1+\sin(\arcsin\sqrt{1-t^2})}$$

$$f(t) = \frac{t}{1+\sqrt{1-x^2}} \Rightarrow f(x) = \frac{x}{1+\sqrt{1-x^2}}$$

(6) If the continuous function $f: [1,4] \to \mathbb{R}$ satisfies the equation $\int_1^2 f(x)\,dx = \int_2^4 [3 - f(x)]\,dx$ then what is the average value of f in the interval $[1,4]$?

$$\int_1^2 f(x)\,dx = \int_2^4 [3-f(x)]\,dx = \int_2^4 3\,dx - \int_2^4 f(x)\,dx$$

$$\Rightarrow \int_1^2 f(x)\,dx + \int_2^4 f(x)\,dx = \int_2^4 3\,dx \Rightarrow \int_1^4 f(x)\,dx = 3x\Big|_2^4 = 3 \cdot 2 = 6$$

$$\bar{f} = \frac{1}{4-1} \cdot \int_1^4 f(x)\,dx = \frac{1}{3} \cdot 6 = 2$$

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(7) If a continuously differentiable function F is defined by $F(x)=\frac{1}{x}\int_{2x}^{x^3-4}[2t-3F'(t-2)]\,dt$ for $x\neq 0$, $F'(2)=?$

$$F'(x)=\frac{-1}{x^2}\cdot\int_{2x}^{x^3-4}[2t-3F'(t-2)]\,dt+\frac{1}{x}\left[3x^2\cdot\left(2(x^3-4)-3F'(x^3-6)\right)-2\cdot\left(4x-3F'(2x-2)\right)\right]$$

$$F'(2)=-\frac{1}{4}\int_{4}^{4}[2t-3F'(t-2)]\,dt+\frac{1}{2}\left[12\left(8-3F'(2)\right)-2\left(8-3F'(2)\right)\right]$$

$$F'(2)=48-18F'(2)-8+3F'(2)\Rightarrow 16F'(2)=40\Rightarrow F'(2)=\frac{5}{2}$$

(8) $\int_{-1}^{1}\frac{x^2\,dx}{1+e^x}=?$

$$(e^x+1)^{-1}=u \qquad x^2dx=dv$$
$$-1(e^x+1)^{-2}dx=du \qquad \frac{x^3}{3}=v$$

$$I=\underbrace{\frac{x^3}{3}\cdot(e^x+1)^{-1}\Big|_{-1}^{1}}_{A}+\underbrace{\int_{-1}^{1}\frac{x^3}{3}\cdot\frac{e^x}{(e^x+1)^2}dx}_{B}=A+B$$

$$A=\frac{1}{3}\cdot\left[\frac{1}{e+1}+\frac{1}{\frac{1}{e}+1}\right]=\frac{1}{3}\left[\frac{1}{e+1}+\frac{e}{e+1}\right]=\frac{1}{3}$$

$$B=\frac{1}{3}\int_{-1}^{1}\frac{x^3\cdot e^x}{(e^x+1)^2}dx \qquad f(x)=\frac{x^3\cdot e^x}{(e^x+1)^2}$$

$$f(-x)=\frac{-x^3\cdot e^{-x}}{(e^{-x}+1)^2}=\frac{-x^3\cdot e^{-x}}{e^{-2x}+2e^{-x}+1}=\frac{-x^3\cdot e^{-x}}{e^{-2x}(1+2e^x+e^{2x})}=\frac{-x^3\cdot e^x}{(e^x+1)^2}=-f(x)$$

$\Rightarrow f(x)$ is an odd function $\Rightarrow B=0$

$$I=\frac{1}{3}+0=\frac{1}{3}$$

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9 Find the area of the region bounded by the curve $y=x^2+x-2$ and the line $y=1-x$ where $x\geq 0$, as unit square.

Intersection points: $x^2+x-2=1-x \Rightarrow x^2+2x-3=0 \Rightarrow x=-3,1$

$x \cdot x = x^2$, $3 \cdot (-1) = -3$, $x - 3 = 2x$

Which one is above and which one is below?

$x=0 \Rightarrow$ (curve) $y_1=-2$ and (line) $y_2=1$ above: line, below: curve

$$A=\int_0^1 [(1-x)-(x^2+x-2)]\,dx=\int_0^1 (3-2x-x^2)\,dx=3x-x^2-\frac{x^3}{3}\Big|_0^1=\frac{5}{3}$$

given as $x\geq 0$

10 If two functions $f,g:\mathbb{R}\to\mathbb{R}$ have the properties $f(x)=1-f(\pi-x)$, $g(x)=g(\pi-x)$ and $\int_0^\pi g(x)\,dx=2$, then $\int_0^\pi f(x)\cdot g(x)\,dx=?$

$$I=\int_0^\pi f(x)\cdot g(x)\,dx=\int_0^\pi [1-f(\pi-x)]\cdot g(x)\,dx=\int_0^\pi g(x)\,dx+\int_0^\pi f(\pi-x)\cdot g(x)\,dx$$

$f(x) \to 1-f(\pi-x)$; $\int_0^\pi g(x)\,dx = 2$; $g(x) = g(\pi-x)$

$\pi - x = t$, $-dx = dt$; $x=0 \Rightarrow t=\pi$, $x=\pi \Rightarrow t=0$

$$I=2+\int_\pi^0 f(t)\cdot g(t)\,dt \quad \text{(swap boundaries)}$$

$$I=2-\int_0^\pi f(t)\cdot g(t)\,dt \Rightarrow 2I=2 \Rightarrow I=1$$

$\int_0^\pi f(t)\cdot g(t)\,dt = I$

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