2024-2025 FALL / MAT1071 MATHEMATICS 1
FINAL EXAM QUESTIONS

(1) If ℓ is a nonzero real number and

$$\lim_{x\to 0}\frac{(1-\cos x)[\sin x-\ln(1+x)]}{x^n}=\ell$$

what is the natural number n?

$$\lim_{x\to 0}\frac{(1-\cos x)[\sin x-\ln(1+x)]}{x^n}=\lim_{x\to 0}\frac{\sin^2 x\,[\sin x-\ln(1+x)]}{x^2\cdot x^{n-2}\cdot(1+\cos x)}$$

($\sin^2 x$: 1, $(1+\cos x)$: 2)

$$=\lim_{x\to 0}\frac{\sin x-\ln(1+x)}{2\cdot x^{n-2}}=\ell\neq 0 \quad (\text{Nominator}\to 0,\ \text{must be indeterminate})$$

$$\overset{L}{=}\lim_{x\to 0}\frac{\cos x-\frac{1}{1+x}}{2\cdot(n-2)\cdot x^{n-3}}=\ell \quad (\text{Again!}) \quad \overset{L}{=}\lim_{x\to 0}\frac{-\sin x+\frac{1}{(1+x^2)}}{2\cdot(n-2)\cdot(n-3)\cdot x^{n-4}}=\ell$$

Nominator $\neq 0 \Rightarrow$ Denominator $\neq 0$ (If 0, limit $\to\infty$)

$$\lim_{x\to 0}x^{n-4}\neq 0 \Rightarrow n-4=0 \Rightarrow n=4$$

(2) If y is a function of x and $x\cdot\cos y=\int_x^y\sqrt{1+t^4}\,dt$, find the equation of the tangent line to the curve $y=f(x)$ at the point $P(0,?)$.

$$x=0 \Rightarrow 0=\int_0^y\sqrt{1+t^4}\,dt \Rightarrow y=0 \Rightarrow P(0,0)$$

$$\cos y-x\cdot\sin y\cdot y'=y'\cdot\sqrt{1+y^4}-\sqrt{1+x^4} \overset{P}{\Longrightarrow} \underbrace{\cos 0}_{1}-0\cdot\sin 0\cdot y'=y'\cdot 1-1$$

$$\Rightarrow 1=y'-1 \Rightarrow m_T=y'=2 \Rightarrow \text{Equation: } y=2x$$

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(3) $\int_{-1}^{1}[\ln(2+x)-\ln(2-x)]\,dx = ?$

$f(x)=\ln(2+x)-\ln(2-x)$

$f(-x)=\ln(2-x)-\ln(2+x)=-f(x) \Rightarrow f(x)$ is odd function.

$$\int_{-1}^{1}[\ln(2+x)-\ln(2-x)]\,dx = 0$$

(4) For a continuous function f, which of the below options for a satisfies the equation $\int_{a}^{x-a} f(t)\,dt = \sin x$?

A) 2 B) 1 C) 0 D) -1 E) -2

Take derivative : $f(x-a)=\cos x$. Let $F'(x)=f(x)$

$\Rightarrow F'(x-a)=\cos x \Rightarrow F(x-a)=\sin x+c$

Also, $\int_{a}^{x-a} f(t)\,dt = \int_{a}^{x-a} F'(t)\,dt = F(t)\Big|_{a}^{x-a} = F(x-a)-F(a)=\sin x$

$x=2a \Rightarrow F(2a-a)=F(a)=\sin 2a+c$

$\sin x+c-\sin 2a-c=\sin x \Rightarrow \sin 2a=0 \Rightarrow a=0$

(5) $\lim_{x\to 0^+}\left[\frac{1}{x}\left(\int_{\pi}^{2\pi} e^{-\sin^2 t}\,dt - \int_{x+\pi}^{2\pi} e^{-\sin^2 t}\,dt\right)\right] = ?$

$$\lim_{x\to 0^+}\frac{\int_{\pi}^{2\pi} e^{-\sin^2 t}\,dt - \int_{x+\pi}^{2\pi} e^{-\sin^2 t}\,dt}{x} \underset{\left(\frac{0}{0}\right)}{\overset{L}{=}} \lim_{x\to 0^+}\frac{e^{-\sin^2(x+\pi)}}{1} = 1$$

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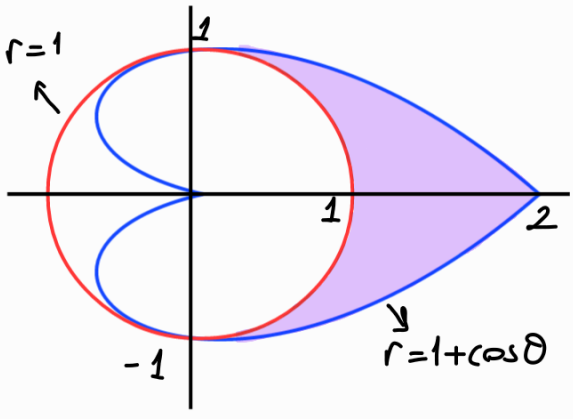
(6) If a continuously differentiable function F is defined for $x \neq 0$, $F(x) = \frac{1}{x} \cdot \int_{2x}^{x^3-4} [2t - 3F'(t-2)]\,dt$, what is the value of $F'(2)$?

$$F'(x) = \frac{-1}{x^2} \cdot \int_{2x}^{x^3-4} [2t - 3F'(t-2)]\,dt + \frac{1}{x}\left[3x^2 \cdot \left(2(x^3-4) - 3F'(x^3-6)\right) - 2 \cdot \left(4x - 3F'(2x-2)\right)\right]$$

$$F'(2) = -\frac{1}{4} \int_{4}^{4} [2t - 3F'(t-2)]\,dt + \frac{1}{2}\left[12\left(8 - 3F'(2)\right) - 2\left(8 - 3F'(2)\right)\right]$$

$$F'(2) = 48 - 18F'(2) - 8 + 3F'(2) \Rightarrow 16F'(2) = 40 \Rightarrow F'(2) = \frac{5}{2}$$

(7) Find the area of the region inside of the cardioid $r = 1 + \cos\theta$ and outside of the circle $r = 1$ in unit square.



$$A = \frac{1}{2} \cdot 2 \int_0^{\pi/2} \left[\underbrace{(1+\cos\theta)^2}_{1+2\cos\theta+\cos^2\theta} - 1\right] d\theta$$

symmetry

$$= \int_0^{\pi/2} \left(2\cos\theta + \frac{1+\cos(2\theta)}{2}\right) d\theta$$

$$= 2\sin\theta + \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\Big|_0^{\pi/2} = 2 \cdot 1 + \frac{\pi}{4} = 2 + \frac{\pi}{4}$$

(8) If a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation $\int_0^{x^2} f(t)\,dt = x^2 \cdot e^{x^2}$, what is the value of $f(-1)$?

Take derivative : $2x \cdot f(x^2) = 2x \cdot e^{x^2} + x^2 \cdot 2x \cdot e^{x^2} \Rightarrow f(x^2) = e^{x^2} + x^2 \cdot e^{x^2}$

$$\Rightarrow f(x) = e^x + x \cdot e^x \Rightarrow f(-1) = e^{-1} - e^{-1} = 0$$

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(9) If the continuous function $f:[0,\infty)\to\mathbb{R}$ satisfies the equation $\int_0^x f(t)dt = x+\int_x^1 t^2 f(t)dt$, what is the result of the integral $\int_0^\infty f(x)dx$?

Take derivative: $f(x)=1-x^2f(x) \Rightarrow f(x)=\frac{1}{1+x^2}$

$$\int_0^\infty f(x)dx = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty} \arctan x\Big|_0^b = \lim_{b\to\infty}(\underbrace{\arctan b}_{\frac{\pi}{2}} - \underbrace{\arctan 0}_{0}) = \frac{\pi}{2}$$

(10) For $x>0$, if $f(x)=\int_1^x \frac{\ln t}{1+t}dt$ and $F(x)=f(x)+f\left(\frac{1}{x}\right)$, which of the following gives F as a function of x?

A) $\ln x$ B) $x-1$ C) $\frac{1}{2}\ln^2 x$ D) x^2 E) x

$$f'(x)=\frac{\ln x}{1+x} \Rightarrow f'\left(\frac{1}{x}\right)=\frac{\ln x^{-1}}{1+\frac{1}{x}}=\frac{-x\cdot\ln x}{1+x}$$

$$F'(x)=f'(x)-\frac{1}{x^2}\cdot f'\left(\frac{1}{x}\right)=\frac{\ln x}{1+x}+\frac{1}{x^2}\cdot x\cdot\frac{\ln x}{1+x}=\frac{x\cdot\ln x+\ln x}{x\cdot(1+x)}$$

$$=\frac{\ln x(1+x)}{x\cdot(1+x)}=\frac{\ln x}{x} \Rightarrow F(x)=\int\frac{\ln x}{x}dx \qquad \ln x=t,\ \frac{dx}{x}=dt$$

$$=\int t\,dt=\frac{t^2}{2}+c=\frac{1}{2}\ln^2 x+c$$

$$F(1)=f(1)+f(1)=0+0=0 \Rightarrow F(x)=\frac{1}{2}\ln^2 x$$

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(11) $\int_{-2}^{2} |1-x^2|\,dx = ?$

	-1		1	
$1-x^2$	−	+		−

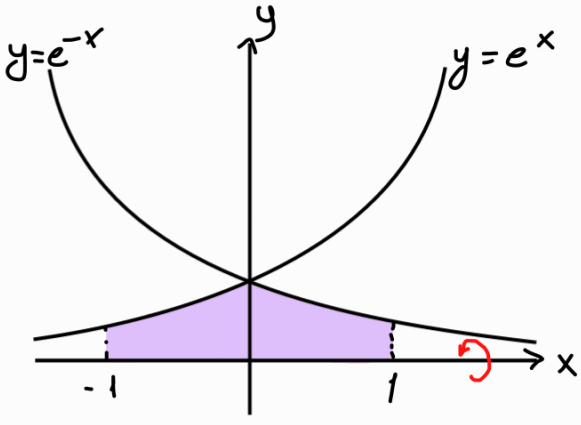
$$I = \int_{-2}^{-1} (x^2-1)\,dx + \int_{-1}^{1} (1-x^2)\,dx + \int_{1}^{2} (x^2-1)\,dx$$

Note that x^2-1 is even function and intervals $[-2,-1]$ and $[1,2]$ have symmetry. Also $[-1,1] \to [0,1]$.2

$$I = 2\cdot\int_{0}^{1} (1-x^2)\,dx + 2\cdot\int_{1}^{2} (x^2-1)\,dx = 2\left(x-\frac{x^3}{3}\right)\Big|_0^1 + 2\cdot\left(\frac{x^3}{3}-x\right)\Big|_1^2$$

$$= 2\cdot\left(1-\frac{1}{3}\right) + 2\left(\frac{8}{3}-2-\frac{1}{3}+1\right) = 2\cdot\frac{6}{3} = 4$$

(12) For $x \in [-1,1]$, if the region bounded by the curves $y=e^x$, $y=e^{-x}$ and the line $y=0$ is rotated about the x-axis, what is the volume of the obtained solid in unit cube?



Disk

$$V = \pi\int_{-1}^{0} e^{2x}\,dx + \pi\int_{0}^{1} e^{-2x}\,dx \quad \text{OR}$$

$$\text{symmetry}: V = 2\pi\int_{0}^{1} e^{-2x}\,dx = 2\pi\,\frac{e^{-2x}}{-2}\Big|_0^1$$

$$= \pi\left(\frac{-1}{e^2}+1\right) = \pi\cdot\frac{e^2-1}{e^2}$$

(13) Which of the following is the angle between the plane $x+2y+3z=5$ and the line $\vec{r}(t) = \langle 1+2t, 3+4t, 6t\rangle$?

A) π B) $\frac{\pi}{2}$ C) $\frac{\pi}{3}$ D) $\frac{3\pi}{4}$ E) $\frac{\pi}{5}$

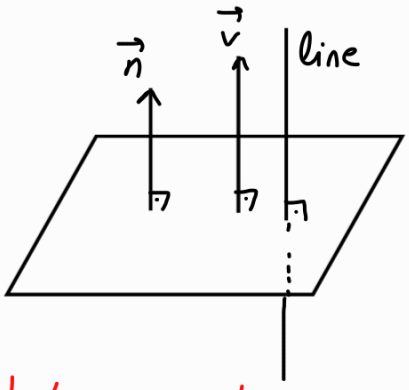
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normal vector of the plane : $\vec{n} = \langle 1,2,3 \rangle$

direction vector of the line : $\vec{v} = \langle 2,4,6 \rangle$

$\vec{v} = 2 \cdot \vec{n} \Rightarrow \vec{v}$ and $\vec{n}$ are parallel.

$\Rightarrow$ line is verticle to the plane



If $\frac{3\pi}{2}$ was an option instead of $\frac{\pi}{2}$, it would be correct.

14 $\int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = ?$

$$\frac{\sqrt{\cos x}+\sqrt{\sin x}-\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} = \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} - \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}} = 1 - \frac{\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}} = \underbrace{\int_0^{\pi/2} 1\,dx}_{x\Big|_0^{\pi/2} = \frac{\pi}{2}} - \underbrace{\int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}}}_{J}$$

For J : $\sin x = \cos\left(\frac{\pi}{2}-x\right)$, $\cos x = \sin\left(\frac{\pi}{2}-x\right)$

$$J = \int_0^{\pi/2} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}\,dx}{\sqrt{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}$$

$\frac{\pi}{2}-x = t$ $\quad x=0 \Rightarrow t=\frac{\pi}{2}$

$-dx = dt$ $\quad x=\frac{\pi}{2} \Rightarrow t=0$

$$J = -\int_{\pi/2}^{0} \frac{\sqrt{\cos t}\,dt}{\sqrt{\sin t}+\sqrt{\cos t}}$$

$$J = +\int_0^{\pi/2} \frac{\sqrt{\cos t}\,dt}{\sqrt{\cos t}+\sqrt{\sin t}} = I \Rightarrow I = \frac{\pi}{2} - I \Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

15 $\int \frac{2x \cdot e^{2x}}{(x+1)^3}\,dx = ?$

$$\frac{(2x+2-2)\cdot e^{2x}}{(x+1)^3} = \frac{2(x+1)\cdot e^{2x}}{(x+1)\cdot(x+1)^2} - \frac{2\cdot e^{2x}}{(x+1)^3} \Rightarrow I = \underbrace{\int \frac{2e^{2x}\,dx}{(x+1)^2}}_{J} - \underbrace{\int \frac{2e^{2x}\,dx}{(x+1)^3}}_{K}$$

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For J: $\frac{1}{(x+1)^2}=u$ $\quad 2\cdot e^{2x}dx=dv$ $\quad J=\frac{e^{2x}}{(x+1)^2}+\int\frac{2e^{2x}dx}{(x+1)^3}$

$\frac{-2\,dx}{(x+1)^3}=du$ $\quad e^{2x}=v$

$$I=J-K=\frac{e^{2x}}{(x+1)^2}+\int\frac{2e^{2x}dx}{(x+1)^3}-\int\frac{2e^{2x}dx}{(x+1)^3}=\frac{e^{2x}}{(x+1)^2}+C$$

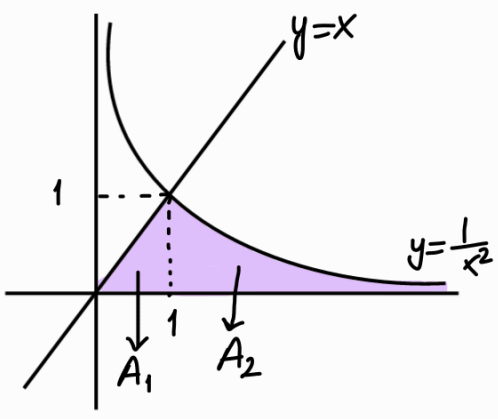
16 If for a continuous function $f:\mathbb{R}\to\mathbb{R}$, $\int_0^2 f(x)dx=2$ and $\int_{-1}^{2} x\cdot f(2x^2)dx=\frac{15}{2}$, what is the value of $\int_0^4 f(2x)dx$?

$\int_0^2 f(x)dx=2$ $\quad x=2t$, $dx=2dt$ $\quad x=0\Rightarrow t=0$, $x=2\Rightarrow t=1$ $\quad \int_0^1 2f(2t)dt=2\Rightarrow\int_0^1 f(2t)dt=1=A$

$\int_{-1}^{2} x\cdot f(2x^2)dx=\frac{15}{2}$ $\quad x^2=t$, $2x\,dx=dt$ $\quad x=-1\Rightarrow t=1$, $x=2\Rightarrow t=4$ $\quad \int_1^4 \frac{f(2t)}{2}dt=\frac{15}{2}\Rightarrow\int_1^4 f(2t)dt=15=B$

$$\int_0^4 f(2x)dx=\int_0^1 f(2x)dx+\int_1^4 f(2x)dx=A+B=16$$

17 What is the area of the region bounded by the lines $y=0$, $y=x$ and the curve $y=\frac{1}{x^2}$ in unit square?



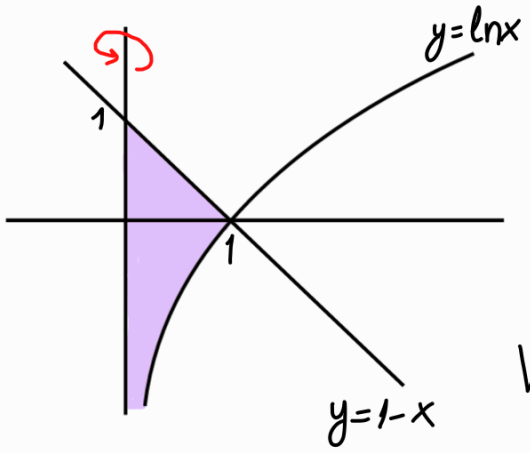
$x=\frac{1}{x^2}\Rightarrow x=1$ (intersection) $\quad A_1=\frac{1\cdot 1}{2}=\frac{1}{2}$ (triangle)

$$A_2=\int_1^\infty\frac{dx}{x^2}=\lim_{b\to\infty}\int_1^b\frac{dx}{x^2}=\lim_{b\to\infty}\frac{-1}{x}\Big|_1^b$$

$$=\lim_{b\to\infty}\left(\underbrace{-\frac{1}{b}}_{\to 0}+1\right)=1\Rightarrow A=A_1+A_2=\frac{1}{2}+1=\frac{3}{2}$$

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18) If the region bounded by the lines $x=0$, $y=1-x$ and the curve $y=\ln x$ is rotated about the y-axis, what is the volume of this obtained solid in unit cube?



$$\text{Shell}: V=2\pi\int_0^1 \underbrace{x\cdot[(1-x)-\ln x]}_{x-x^2-x\ln x}\,dx$$

improper! (for $\ln x$)

$$V=2\pi\left[\int_0^1 (x-x^2)\,dx-\lim_{a\to 0}\int_a^1 x\ln x\,dx\right]$$

$\ln x=u \quad x\,dx=dv$

$\frac{dx}{x}=du \quad \frac{x^2}{2}=v$

$$V=2\pi\left[\underbrace{\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1}_{\frac{1}{2}-\frac{1}{3}=\frac{1}{6}}-\lim_{a\to 0}\left[\frac{x^2}{2}\cdot\ln x-\underbrace{\int\frac{x^2}{2}\cdot\frac{dx}{x}}_{\frac{x^2}{4}}\right]_a^1\right]$$

$$V=2\pi\left[\frac{1}{6}-\lim_{a\to 0}\left(\underbrace{\frac{1}{2}\cdot\ln 1}_{0}-\frac{1}{4}-\underbrace{\frac{a^2}{2}\ln a}+\underbrace{\frac{a^2}{4}}_{0}\right)\right]=2\pi\left(\underbrace{\frac{1}{6}+\frac{1}{4}}_{\frac{5}{12}}\right)=\boxed{\frac{5\pi}{6}}$$

$$\lim_{a\to 0} a^2\cdot\ln a \overset{(0\cdot\infty)}{=} \lim_{a\to 0}\frac{\ln a}{\frac{1}{a^2}} \overset{(\frac{\infty}{\infty})}{\underset{\text{L}}{=}} \lim_{a\to 0}\frac{\frac{1}{a}}{\frac{-2}{a^3}}=0$$

2nd way (Disk): $V=\pi\left[\int_{-\infty}^0 e^{2y}\,dy+\int_0^1 \underbrace{(1-y)^2}_{1-2y+y^2}\,dy\right]$

$$V=\pi\left[\lim_{a\to-\infty}\underbrace{\int_a^0 e^{2y}\,dy}+\left(y-y^2+\frac{y^3}{3}\right)\Big|_0^1\right]=\pi\left(\frac{1}{2}+1-1+\frac{1}{3}\right)=\boxed{\frac{5\pi}{6}}$$

$$\frac{e^{2y}}{2}\Big|_a^0=\frac{1}{2}-\underbrace{\frac{e^{2a}}{2}}_{\to 0}$$

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19) Which of the following is a curve whose arc length integral is $s=\int_0^{\pi}\sqrt{1+4x^2\cdot\sin^2(x^2)}\,dx$ and passes through the point $P(\sqrt{\pi},1)$?

A) $y=1$ B) $y=1+\sqrt{\pi}-x$ C) $y=-\cos x^2$ D) $y=3+2\cos x^2$ E) $y=1+\sin x^2$

$$(y')^2=4x^2\cdot\sin^2x^2 \Rightarrow y'=2x\sin x^2 \Rightarrow y=\int 2x\sin x^2\,dx \qquad \begin{matrix}x^2=t\\ 2x\,dx=dt\end{matrix}$$

$$y=\int\sin t\,dt=-\cos t+C=-\cos x^2+C \Rightarrow \underbrace{-\cos\pi}_{\begin{matrix}-1\\ 1\end{matrix}}+C=1 \Rightarrow C=0$$

$$y=-\cos x^2$$

20) For $x>0$, what is the partial fraction decomposition of the integral $\int\frac{2dx}{x(\ln^2x-\ln x^2)}$?

$$\frac{2}{\ln^2x-2\ln x}\xrightarrow{\ln x=t}\frac{2}{t^2-2t}=\frac{2}{t(t-2)}=\frac{A}{t}+\frac{B}{t-2}$$

$$A=\frac{2}{t-2}\Big|_{t=0}=-1\,,\quad B=\frac{2}{t}\Big|_{t=2}=1 \Rightarrow \frac{1}{\ln x-2}-\frac{1}{\ln x}$$

$$\Rightarrow \int\left(\frac{1}{x(\ln x-2)}-\frac{1}{x\ln x}\right)dx=\int\left(\frac{1}{x\ln x-2x}-\frac{1}{x\ln x}\right)dx$$

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